Advanced Yield Curve Calibration, Mixed Interpolation Schemes

& How to Incorporate Jumps and the Turn-of-Year Effect

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Abstract

Yield curves are used to imply the forward rates and discount factors from market tradable instruments and are required to discount future cash flows and evaluate the price of all financial contracts. Not all instruments can be included in the yield curve calibration or fitting process, hence we interpolate any gaps and missing forward rates. In this paper we discuss interpolation best practise and how to incorporate market jumps and turn of year (ToY) effects into yield curve calibration.

Advanced Curves

Advanced yield curves are required to be highly accurate to typically to 1/10th of a basis point[1] at a minimum if they are to be considered suitable for trading purposes. To reach this high standard of accuracy it is important that yield curves fit market dynamics well with a suitable interpolation scheme as determined by market instruments.

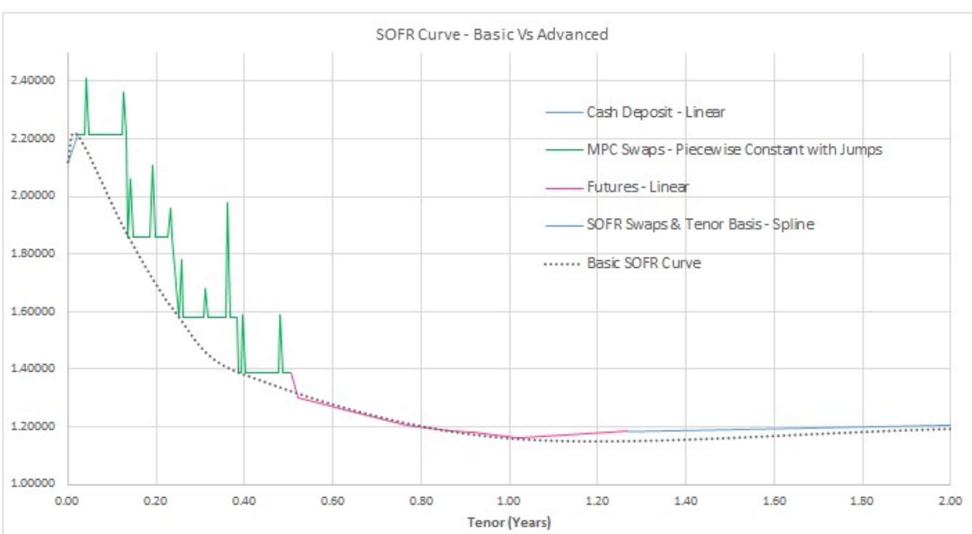

Figure 1: USD SOFR Yield Curve with Mixed Interpolation

[1] A basis point (bps) is 1/100th of a percent i.e. 0.01%

Hybrid or Mixed Interpolation

To calibrate a yield curve to match market prices, mixed interpolation and careful consideration of instrument joins and overlaps is required. Additionally one has to cater for the turn-of-year effect and jumps in yield curve forward rates. Basic yield curves without this ability will result in swap quotes being off-market, as illustrated in figure (1) where we compare a basic and advanced USD SOFR[2] yield curve.

Interpolation Best Practise by Calibration Instrument

- Cash Deposits — Linear Interpolation
- Monetary Policy Swaps[3] — Piecewise Constant (flat) with Jumps
- Futures — Linear Interpolation
- Swaps — Smooth Interpolation e.g. spline

Forward Rate Turns and Jumps

A key consideration when building and calibrating yield curves is how to incorporate the potentially large spikes in forward rates in areas of the curve where there is a shortage of liquidity or squeeze in the market, which is often the case around quarter or year end when many financial institutions refinance their trading positions at the same time, see figure (2).

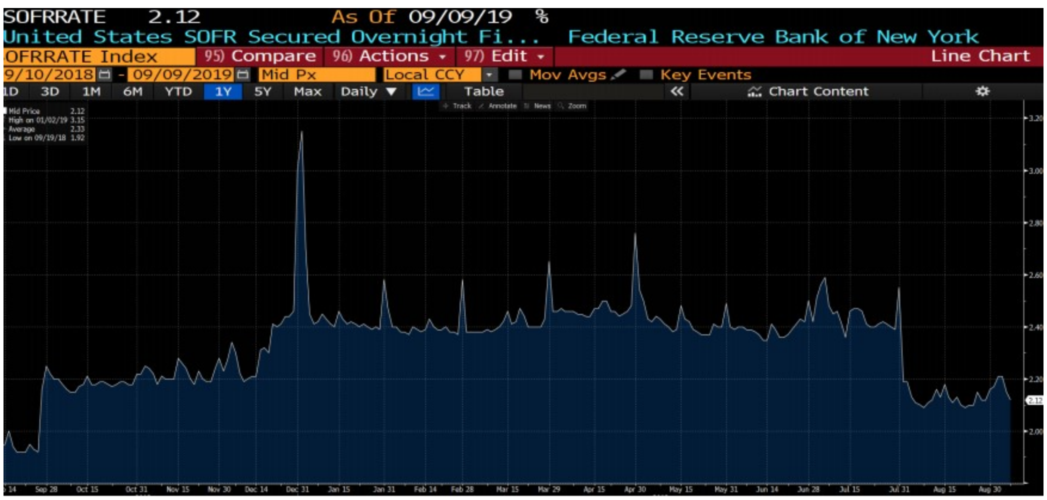

Figure 2: USD SOFR Forward Rate Jumps and Turns

Discount Factor and Forward Rate Formulae

For a given curve the discount factor P(t,T) at payment date T is defined as,

$$P(t,T) = exp\left(-\int_{u=t}^{T} f(t,u)\,du\right) \tag{1}$$

[2] Secured Overnight Funding Rate (SOFR) based on US Treasury repurchase market transactions and published by the Federal Reserve Bank of New York (FED).

[3] Also known as MPC swaps; examples include USD FOMC swaps and EUR ECB swaps which are short-dated single period swaps between central bank meeting and rate setting dates.

By rearranging equation (1) gives the instantaneous forward rate f(t,T) as,

$$f(t,T) = -\frac{\partial}{\partial T}\ln(\,P(t,T)\,) \tag{2}$$

where t denotes the valuation date, T the cash flow payment date and f(t,T) is the forward rate at time T.

<u>Yield Curves: Incorporating Jumps and Turn-of-Year (ToY) Effects</u>
We can adjust forward rate and discount formulae to incorporate jumps and turns by incorporating an overlay curve, which is tantamount to applying a spread ε to the forward rate on jump or turn date τ as illustrated in figure (3).

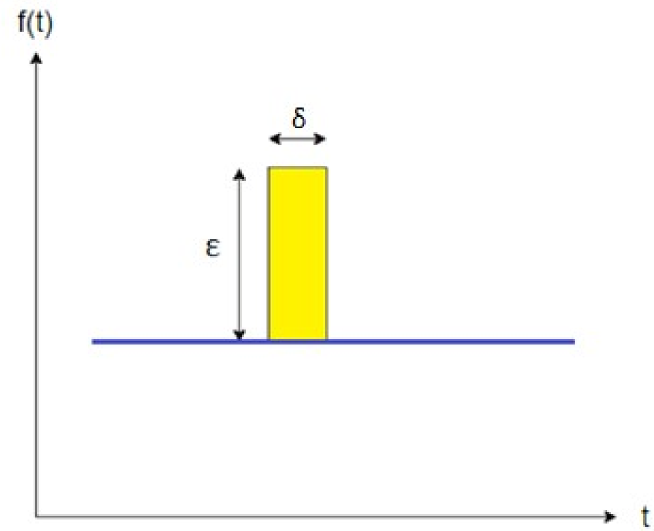

Figure 3: Flat Forward Rate Curve with a Jump

If we consider a single daily jump ε at time τ over a time period δ of 1 day, then the adjusted forward rate f*(t,T) at time T can be modelled as,

$$f^*(t,T) = f(t,T) + \varepsilon\,.\,\mathbf{1}\{\,\boldsymbol{\tau} = \boldsymbol{T}\,\} \tag{3}$$

where $\mathbf{1}\{\,\boldsymbol{\tau} = \boldsymbol{T}\,\}$ is an indicator function that is equal to 1 if the jump at time τ occurs on the forward date T and is zero otherwise. Similarly $\mathbf{1}\{\,\boldsymbol{\tau} \neq \boldsymbol{T}\,\}$ is the indicator function that is equal to 1 if the jump at time τ does not occur on the forward date T and is zero otherwise.

Similarly adjusted discount factors P*(t,T) can be calculated as below,

$$P^*(t,T) = exp\left(-\int_{u=t}^{T} f(t,u)\,du - \varepsilon\,\delta\,.\,\mathbf{1}\{\,\boldsymbol{t} \leq \boldsymbol{\tau} \leq \boldsymbol{T}\,\}\right)$$

The adjusted discount formula must accumulate all jumps and turns within the discount factor interval,

$$P^*(t,T) = \exp\left(-\int_{u=t}^{T} f(t,u)du\right) + \exp(-\varepsilon\,\delta)\ .\mathbf{1}\underbrace{\{\boldsymbol{t} \le \boldsymbol{\tau} \le \boldsymbol{T}\}}_{\substack{\textit{1 if a jump}\\ \textit{0 if no jump}}}$$

giving,

$$P^*(t,T) = \underbrace{P(t,T)}_{\substack{\textit{Unadjusted}\\ \textit{Discount Factor}}} \underbrace{+\exp(-\varepsilon\,\delta)\ .\mathbf{1}\{\boldsymbol{t} \le \boldsymbol{\tau} \le \boldsymbol{T}\}}_{\substack{\textit{Jump or}\\ \textit{ToY Adjustment}}} \qquad (4)$$

when τ does lie in the interval [t, T] equation (4) gives,

$$\begin{aligned} P^*(t,T) &= P(t,T) + \exp(-\varepsilon\,\delta)\ .\mathbf{0} \\ &= P(t,T) \end{aligned}$$

and when τ is inside the interval [t, T] we have,

$$\begin{aligned} P^*(t,T) &= P(t,T) + \exp(-\varepsilon\,\delta)\ .\mathbf{1} \\ &= P(t,T) + \exp(-\varepsilon\,\delta) \end{aligned}$$

Finally in the generic case if we consider a table or series of n daily jumps and turns we can modify equation (3) and (4) to incorporate these jumps as follows,

$$f^*(t,T) = f(t,T) + \sum_{i=1}^{n} \varepsilon_i\ .\ \mathbf{1}\{\boldsymbol{\tau_i} = \boldsymbol{T}\} \qquad (5)$$

and

$$P^*(t,T) = P(t,T) + \prod_{i=1}^{n} \exp(-\,\varepsilon_i\delta)\ .\ \mathbf{1}\{\boldsymbol{t} \le \boldsymbol{\tau_i} \le \boldsymbol{T}\} \qquad (6)$$

<u>Turn-of-Year Example:</u>

Consider the following turn table of daily turn points,

Turn, ε	***Time, τ***
1.00%	2.0
0.75%	3.0
0.25%	4.0

Firstly let's evaluate adjusted six month forward rates at fixing times t = 1, 2, 3, 4, 5 with T = t + 0.5 (i.e. six months later).

Using equation (5) we have,

$f^*(1, 1.5) = f(1, 1.5) + (1.00\% \times 0 + 0.75\% \times 0 + 0.25\% \times 0) = f(0,1)$
$f^*(2, 2.5) = f(2, 2.5) + (\mathbf{1.00\% \times 1} + 0.75\% \times 0 + 0.25\% \times 0) = f(0,2) \mathbf{+ 1.00\%}$
$f^*(3, 3.5) = f(3, 3.5) + (1.00\% \times 0 + \mathbf{0.75\% \times 1} + 0.25\% \times 0) = f(0,3) \mathbf{+ 0.75\%}$
$f^*(4, 4.5) = f(4, 4.5) + (1.00\% \times 0 + 0.75\% \times 0 + \mathbf{0.25\% \times 1}) = f(0,4) \mathbf{+ 0.25\%}$
$f^*(5, 5.5) = f(5, 5.5) + (1.00\% \times 0 + 0.75\% \times 0 + 0.25\% \times 0) = f(0,5)$

Secondly Using equation (6) the adjusted discount factors forwards with valuation date t = 0 for maturity dates T = 1, 2, 3, 4 and 5 we have,

$P^*(0,1) = P(0,1) + \exp(-1.00\%\ \delta) \times 0 + \exp(-0.75\%\ \delta) \times 0 + \exp(-0.25\%\ \delta) \times 0$
$= P(0,1)$

$P^*(0,2) = P(0,2) + \mathbf{\exp(-1.00\%\ \delta) \times 1} + \exp(-0.75\%\ \delta) \times 0 + \exp(-0.25\%\ \delta) \times 0$
$= P(0,2) + \exp(-1.00\%\ \delta)$

$P^*(0,3) = P(0,3) + \mathbf{\exp(-1.00\%\ \delta) \times 1} + \mathbf{\exp(-0.75\%\ \delta) \times 1} + \exp(-0.25\%\ \delta) \times 0$
$= P(0,3) + \exp(-1.00\%\ \delta) + \exp(-0.75\%\ \delta)$

$P^*(0,4) = P(0,4) + \mathbf{\exp(-1.00\%\ \delta) \times 1 + \exp(-0.75\%\ \delta) \times 1 + \exp(-0.25\%\ \delta) \times 1}$
$= P(0,4) + \exp(-1.00\%\ \delta) + \exp(-0.75\%\ \delta) + \exp(-0.25\%\ \delta)$

$P^*(0,5) = P(0,5) + \mathbf{\exp(-1.00\%\ \delta) \times 1 + \exp(-0.75\%\ \delta) \times 1 + \exp(-0.25\%\ \delta) \times 1}$
$= P(0,5) + \exp(-1.00\%\ \delta) + \exp(-0.75\%\ \delta) + \exp(-0.25\%\ \delta)$

As we can see from the results forwards rates fixing on turn dates are adjusted for turns, however discount factors incorporate all turns within the discount factor interval.

<u>Implementation</u>
In practise turns are commonly implemented using overlay curves, whereby we construct a regular yield curve with no turns and model a stand-alone turn curve that is zero everywhere except at jump dates where it has a value equal to the jump size as shown below in figure (4) and figure (5).

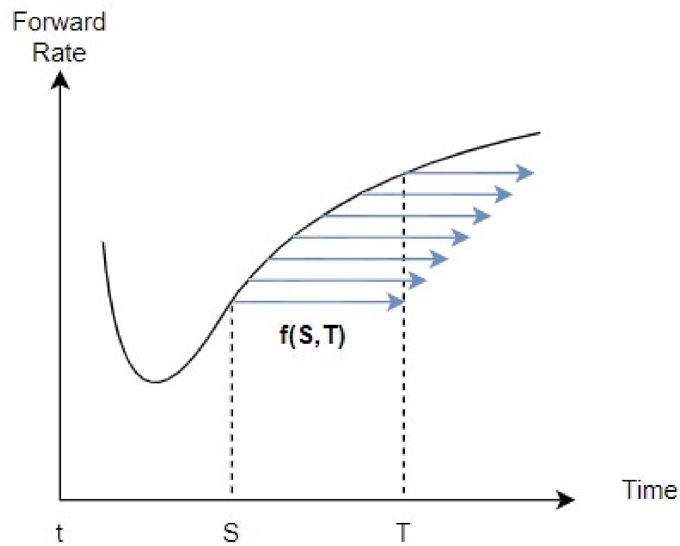

Figure 4: Forward Curve Illustration

> ***Forward Curve:*** *The forward rate f(S,T) from time S to T for a USD 3M LIBOR rate say would have an interval length [S,T] of 3 months. The blue arrows indicate other instantaneous USD 3M LIBOR forward rates all starting on the forward curve and ending 3 months later. The area under the forward curve corresponds to the discount factor, where for example P(S,T) is the discount factor from S to T and corresponds to the area under the curve from time S to T.*

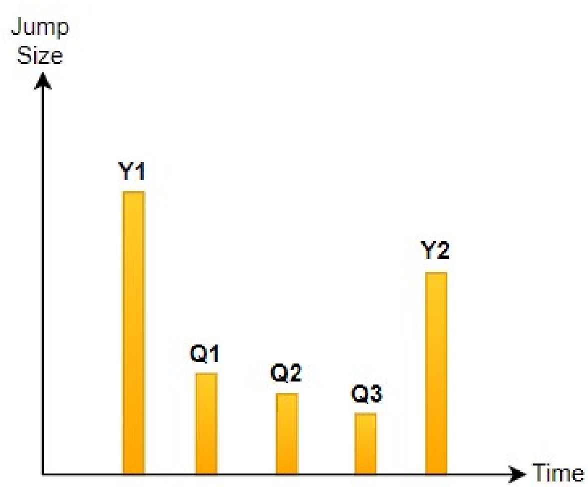

Figure 5: *Overlay Jump Curve Illustration*

Y1 *– First Year End,* ***Y2*** *= Second Year End*
Q1 *= First Quarter End,* ***Q2*** *= Second Quarter End,* ***Q3*** *= Third Quarter End*

> **Overlay Jump Curve:** The jump curve has zero value everywhere except for the points where a jump or turn has been specified, which are typically around year-end and quarter-end points for the first one to three years on the short end of the curve.

Conveniently when using overlay curves the adjusted forward rate and discount factors from equation (5) and equation (6) become simple additive expressions as follows,

$$f^*(t,T) = f(t,T) + f_{Jump}(t,T) \tag{7}$$

and,

$$P^*(t,T) = P(t,T) + P_{Jump}(t,T) \tag{8}$$

where $f_{Jump}(t,T)$ and $P_{Jump}(t,T)$ represent the forward rate and discount factor from the jump overlay curve respectively.

When combined the regular forward and overlay curves give a convenient representation of the adjusted forward curve as illustrated in figure (6) below,

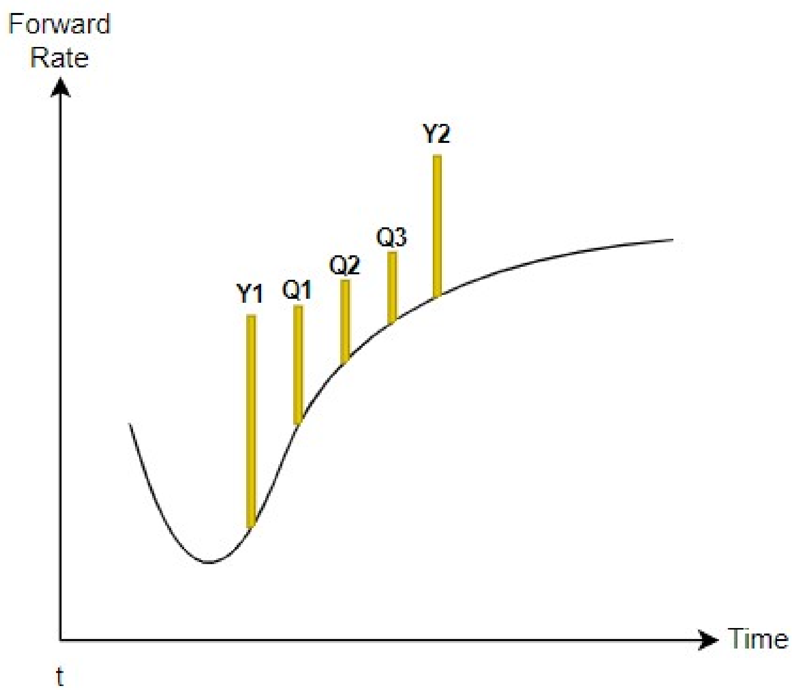

Figure 6: *Adjusted Forward Curve with Discrete Jumps*

Y1 *– First Year End,* ***Y2*** *= Second Year End*
Q1 *= First Quarter End,* ***Q2*** *= Second Quarter End,* ***Q3*** *= Third Quarter End*