Quant Notes - How to Solve & Minimize Complex Equations using the Newton-Raphson Method

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In this short paper we outline the Newton-Raphson methods used for solving and minimizing complex equations that often have no analytical solution. We outline the formulae, their origins and give a simple example of their application.

Definition 1: Newton-Raphson Solver

The **Newton-Raphson** formula to find the root of a **solution** is defined as follows,

$$x_{n+1} = x_n - \frac{f(x)}{f'(x_n)} \tag{1a}$$

often written as,

$$x_{n+1} = x_n - J^{-1} f(x) \tag{1b}$$

Definition 2: Newton-Raphson Minimizer

Likewise the Newton-Raphson formula to find the **minimum** of an equation is,

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} \tag{2a}$$

which could be written as,

$$x_{n+1} = x_n - H^{-1} J \tag{2b}$$

where J and f'(x) denote the **Jacobian** or first order derivative of f(x) and H and f''(x) denote the **Hessian** or second order derivative.

Newton-Raphson Solver – Origins and Usage

The Newton-Raphson Solver formula can be used to solve for the solution of an equation, where it is difficult to do so analytically. It is derived from the equation of the tangent line to y = f(x) at the point (x_0, y_0) with slope m namely,

$$y - y_0 = m(x - x_0) \tag{3a}$$

Alternatively and equivalently this can be written as,

$$y - f(x_0) = f'(x_0)(x - x_0) \tag{3b}$$

The root of the equation and solution is given when the tangent intersects the x-axis where y = 0 and x = x_1 giving,

$$-f(x_0) = f'(x_0)(x_1 - x_0) \tag{4}$$

Rearranging for x_1 gives the recursive Newton-Raphson formula, which requires an **initial guess x_0**,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{5}$$

Using the initial guess we can calculate x_1, we then use x_1 to imply x_2 and repeat the **iterative process** until the x values **converge** to solution, namely when the difference between x values is below a specified tolerance ε namely,

$$|\, x_{n+1} - x_n \,| < \varepsilon \tag{6}$$

The tolerance should be selected to be sufficiently small for best results. The Newton-Raphson iteration process is illustrated in figure 1, showing convergence to the solution at point c.

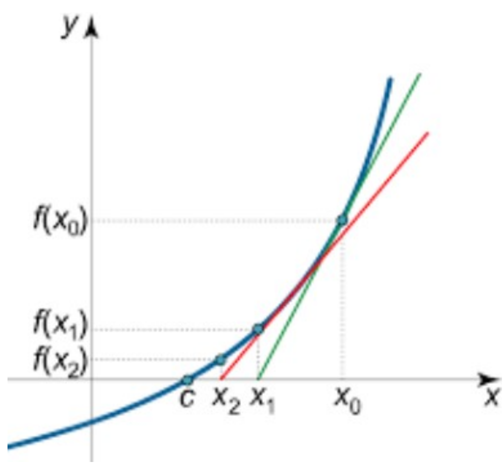

Figure 1: Newton-Raphson Search for Solution, c

<u>Example 1: Solving for a Solution</u>
Given an equation f(x) = $x^3 - x - 1$ an initial guess **x_0 = 5** and tolerance **ε = 0.01**.

Analytically we know we have a real solution with f(x) = 0 when x = 1.32, which we can confirm visually, see figure 2.

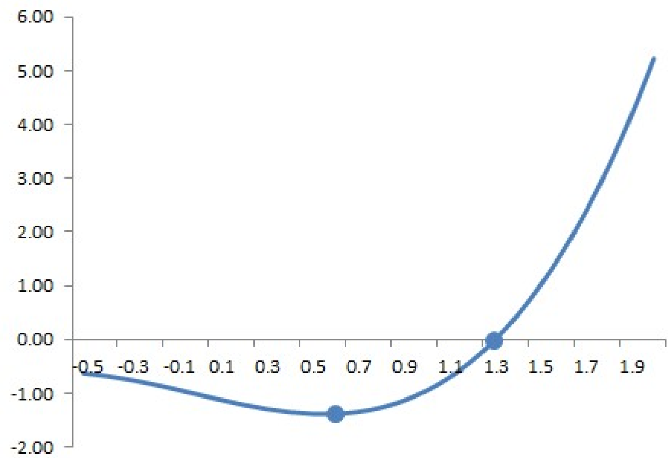

Figure 2: *Graph of $f(x) = x^3 - x - 1$ with a real root at x = 1.32 and a minimum value at x = 0.58*

Let's confirm the solution using Newton-Raphson formula (1a). Firstly we compute the Jacobian or first derivative, which in this case is **$f'(x) = 3x^2 - 1$**. For complex problems we can compute this numerically by bumping i.e. by calculating f(x+dx) – f(x) / dx say.

Secondly we iterate using equation (1a) and the initial guess $x_0 = 5$ until the new value of x converges to within tolerance ε = 0.01. This leads to a solution of x = 1.32 as shown in table 1.

n	x_{n+1}	x_n	$f(x_n)$	$f'(x_n)$	ε
1	3.39	5.00	119.00	74.00	1.608
2	2.36	3.39	34.63	33.51	1.033
3	1.74	2.36	9.76	15.69	0.622
4	1.43	1.74	2.50	8.04	0.311
5	1.33	1.43	0.47	5.10	0.093
6	**1.32**	1.33	0.04	4.33	0.008

Table 1: Newton-Raphson process leading to a solution of x = 1.32 in 6 iterations

Newton-Raphson Minimization

Likewise to iteratively search for the value of x that minimizes f(x) we first set,

$$x_{n+1} = x_n + t \tag{7}$$

Next we assume f(x) is twice differentiable and approximate the function as a **second order Taylor Series expansion** of $f(x_n)$ about a point t.

$$f(x_n + t) \approx f(x_n) + f'(x_n)t + \frac{1}{2}f''(x_n)t^2 \tag{8}$$

The minimum of equation (8) is then found by setting to zero the first derivative with respect to t, that is to say we evaluate where the slope is zero,

$$0 = \frac{d}{dt}\left(f(x_n) + f'(x_n)t + \frac{1}{2}f''(x_n)t^2\right) = f'(x_n) + f''(x_n)t \qquad (9)$$

The minimum is given by,

$$t = -\frac{f'(x_n)}{f''(x_n)} \qquad (10)$$

Substituting (10) into (7) gives the result,

$$x_{n+1} = x - \frac{f'(x_n)}{f''(x_n)} \qquad (11)$$

Example 2: Finding the Minimum of a Function

Given an equation $f(x) = x^3 - x - 1$ an initial guess $\mathbf{x_0 = 5}$ and tolerance $\boldsymbol{\varepsilon} = \mathbf{0.01}$.

Analytically we know the minimum value, f'(x) = 0 when x =0.58 as shown in figure 2. Again let's confirm this using Newton-Raphson formula (2a).

Firstly we compute the Jacobian first and Hessian second derivatives, which in this case is $\mathbf{f'(x) = 3x^2\text{-}1}$ and $\mathbf{f''(x) = 6x}$. Again for complex problems we can compute this numerically by bumping i.e. by calculating f(x+dx) – f(x) / dx say.

Secondly we iterate using equation (2a) and the initial guess $x_0 = 5$ until the new value of x converges to within tolerance ε = 0.01. This leads to a solution of x = 0.58 as shown in table 2.

n	x_{n+1}	x_n	$f(x_n)$	$f'(x_n)$	ε
1	2.53	5.00	74.00	30.00	2.467
2	1.33	2.53	18.25	15.20	1.201
3	0.79	1.33	4.33	7.99	0.541
4	0.61	0.79	0.88	4.75	0.185
5	0.58	0.61	0.10	3.64	0.028
6	**0.58**	0.58	0.00	3.47	0.001

Table 2: Newton-Raphson iteration leading to a solution of x = -3