Solving Complex Equations Subject to Constraints

Using Lagrangian Constrained Optimization

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Abstract

In this paper we outline the Lagrangian constrained optimization method to solve complex problems subject to constraints. Firstly we summarize the Lagrangian constrained optimization routine. Secondly we outline a detailed implementation strategy. Thirdly and finally we provide example and solve a financial problem using the Lagrangian technique.

1. Lagrangian Constrained Optimization

To find the minimum or maximum value of a multivariate equation f(x,y) subject to a constraint g(x,y) this can be achieved using the Lagrangian technique as follows,

Step 1: Create the Lagrange Equation

Set the equation f(x,y) equal to a λ multiple of the constraint g(x,y).

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda\, g(x, y)$$

The λ parameter is called the Lagrange multiplier and gives the rate of change of the solution value per unit constraint.

Step 2: Solve for the Critical Points

Next differentiate the Lagrange equation and set the result to the zero.

$$\nabla\mathcal{L}(\mathrm{x}, \mathrm{y}, \lambda) = \nabla f(x, y) - \lambda\, \nabla g = 0$$

The solution will provide all the turning points where the slope of the function and constraint are zero, which may include local maxima and minima.

Step 3: Evaluate the Solutions

Having found the (x,y,λ) solution value(s) in step 2, we plug each solution into the Lagrange equation from step 1 to determine which gives the maximum and / or minimum.

2. Detailed Lagrangian Implementation

We proceed by giving a more detailed overview of the Lagrangian method, including how to frame the problem to be solved and how configure any constraints.

a) The Problem:

Suppose we want to find the values of x, y that produce the maximum (or minimum) value of a function, f(x,y) subject to the constraint g(x,y) ≤ 0. This can be expressed as follows,

$$\max_{x,y} f(x,y) \ subject\ to\ g(x,y) \ \leq 0 \tag{1a}$$

or,

$$\min_{x,y} f(x,y) \ subject\ to\ g(x,y) \ \leq 0 \tag{1b}$$

b) The Constraints:

In their raw format "**less-than**" constraints are expressed as,

$$x + y \leq a$$

which are rearranged to fit the problem described in equations (1) as,

$$g(x,y) = x + y - a \leq 0 \tag{2a}$$

However for "**greater-than"** constraints specified as,

$$x + y \geq a$$

This leads to,

$$g(x,y) = x + y - a \geq 0 \tag{2b}$$

which we multiply by minus one to match the equation (1) specification namely,

$$-g(x,y) = a - x - y \leq 0$$

Consequently for "**greater-than**" constraints of the format specified in equation (2b) we must make g(x,y) negative in all equations and specifically flip the sign of g(x,y) in equations (1), (3) and (4).

c) The Lagrangian Method:

The Lagrangian method determines the maximum or the minimum as the point where the slope or first derivative of the problem is zero. It requires we assume the function f(x,y) and constraint g(x,y) are smooth, since a gradient approach is being used. We determine if the point is a minimum or maximum by taking the second derivative[1] or simply by evaluating f(x,y) at the solution points.

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[1] A positive (negative) second derivative indicates a minimum (maximum) value.

Given a constant of proportionality λ, also known as the **Lagrangian multiplier** we define the function to be solved as a scalar multiple of the constraint,

$$f(x,y) = \lambda\, g(x,y)$$

We rearrange and define this as the **Lagrangian** $\mathcal{L}(\mathrm{x},\mathrm{y},\lambda)$ as follows,

$$\mathcal{L}(\mathrm{x},\mathrm{y},\lambda) = f(x,y) - \lambda\, g(x,y) \qquad (3)$$

The constrained maximum (or minimum) solution is given by the turning or inflection points (x,y) where the first derivative is zero.

$$\nabla\mathcal{L}(\mathrm{x},\mathrm{y},\lambda) = \nabla f(x,y) - \lambda\, \nabla g(x,y) = \underline{\mathbf{0}} \qquad (4)$$

where ∇ denotes the gradient vector and <u>**0**</u> the zero vector.

Solving the Lagrangian using equation (4) may give multiple solutions. To determine if the solution gives a minimum or maximum value, we evaluate each solution directly or take second derivatives.

d) <u>Multiple Constraints</u>

To apply **multiple constraints** we extend equation (4) with additional constraints h(x,y) and i(x,y) as follows,

$$\mathcal{L}(\mathrm{x},\mathrm{y},\lambda) = \nabla f(x,y) - \lambda_1\, \nabla g(x,y) - \lambda_2\, \nabla h(x,y) - \lambda_3\, \nabla i(x,y) = \underline{\mathbf{0}} \qquad (5)$$

We can add as many constraints as required and the Lagrangian equation will determine the solution, if one exists.

e) <u>The Meaning of λ</u>

The λ parameter is called the Lagrange multiplier and gives the rate of change of the minimum or maximum solution per unit constraint. Increasing (decreasing) the constraint value by 1 unit would increase (decrease) the minimum or maximum value of the function by λ.

3. <u>Lagrangian Example</u>

To conclude we illustrate how to apply the Lagrangian technique to a financial problem[2]. Consider the problem where we want to maximize the revenue of a car factory where revenue R(h,s) is a function of hours of labour, h charged at \$20 per hour and amount of steel required, s charged at \$170 per tonne.

The company financial adviser estimates the revenue function as below and advises factory management accordingly,

$$R(h,s) = 200\, h^{2/3}\, s^{1/3}$$

[2] This example is taken from: https://www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives/constrained-optimization/a/lagrange-multipliers-examples

Management are able to finance a budget, B of $20,000 this financial year. How many hours of labour and how much steel should management provision for in order to maximize their revenue?

Visually this problem looks as below, where the blue line denotes the revenue and the red line the budget constraints.

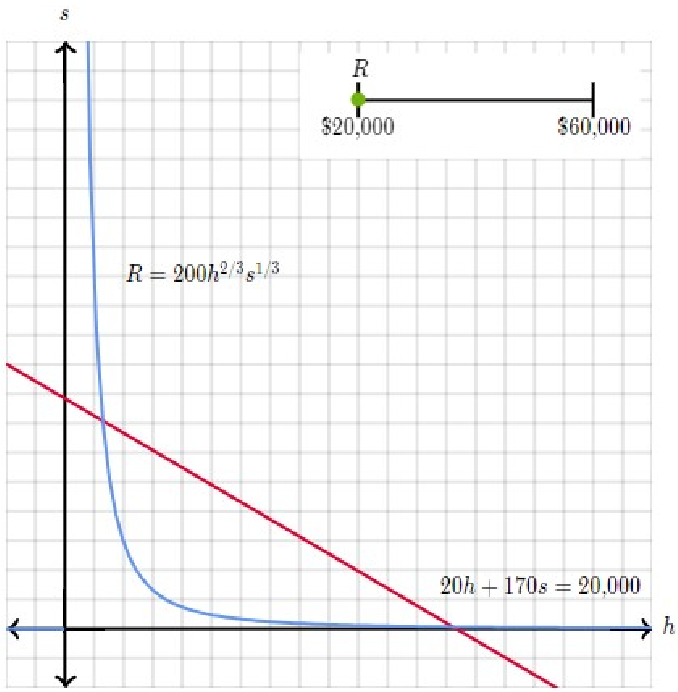

Figure 1: Plot of revenue and budget constraints. (**Source**: Kahn Academy[3])

<u>Prerequisite Step: Set-Up Budget Constraint, B(h,s)</u>

The budget must be less than $20,000 with unit costs of 20h and 170s giving,

$$B(h,s) = 20h + 170s \leq 20{,}000$$

We rearrange the constraint to mimic equation (2),

$$B(h,s) = 20h + 170s - 20{,}000 = 0$$

<u>Step 1: Create the Lagrangian Equation</u>

The Lagrangian equation (3) combines the revenue R(h,s) and the budget constraints B(h,s) as into a single equation as follows,

$$\mathcal{L}(\mathrm{h,s,\lambda}) = R(h,s) - \lambda\, B(h,s)$$

[3] https://www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives/constrained-optimization/a/interpretation-of-lagrange-multipliers

This leads to,

$$\mathcal{L}(\mathrm{h}, \mathrm{s}, \lambda) = 200\, h^{2/3}\, s^{1/3} - \lambda(20h + 170s - 20{,}000)$$

<u>Step 2: Solve the Critical Points</u>

We solve for the minimum and maximum by taking the first derivative and applying equation (4),

$$\nabla\mathcal{L}(\mathrm{h}, \mathrm{s}, \lambda) = \nabla R(h, s) - \lambda\, \nabla B(h, s) = \underline{\mathbf{0}}$$

Evaluating the first derivatives gives,

$$\nabla R = \begin{bmatrix} \partial R / \partial h \\ \partial R / \partial s \end{bmatrix} = \begin{bmatrix} 200.\, {}^{2}/_{3}\, h^{-1/3} \,.\, s^{1/3} \\ 200.\, {}^{1}/_{3}\, h^{2/3} \,.\, s^{-2/3} \end{bmatrix} = \begin{bmatrix} {}^{400}/_{3} \cdot \left(\frac{s}{h}\right)^{1/3} \\ {}^{200}/_{3} \cdot \left(\frac{h}{s}\right)^{2/3} \end{bmatrix}$$

and

$$\nabla B = \begin{bmatrix} \partial B / \partial h \\ \partial B / \partial s \end{bmatrix} = \begin{bmatrix} 20 \\ 170 \end{bmatrix}$$

Putting everything together we establish a system of equations that when solved give us the minimum and/or maximum solution. The constraint is required so that we have full rank i.e. 3 equations are required to find the 3 unknown parameters.

$$\nabla\mathcal{L}(\mathrm{h}, \mathrm{s}, \lambda) = \begin{bmatrix} {}^{400}/_{3} \cdot \left(\frac{s}{h}\right)^{1/3} - 20\lambda \\ {}^{200}/_{3} \cdot \left(\frac{h}{s}\right)^{2/3} - 170\lambda \end{bmatrix} = \underline{\mathbf{0}} \qquad \begin{matrix} (E1) \\ (E2) \end{matrix}$$

and

$$20h + 170s = 20{,}000 \qquad (E3)$$

We proceed to solve this system of equations as follows. Firstly from equation (E1)

$$\lambda = {}^{20}/_{3} \cdot \left(\frac{s}{h}\right)^{1/3} \qquad (E4)$$

which we substitute into (E2),

$${}^{200}/_{3} \cdot \left(\frac{h}{s}\right)^{2/3} - {}^{3{,}400}/_{3} \cdot \left(\frac{s}{h}\right)^{1/3} = 0$$

We then multiply by $\left(\frac{h}{s}\right)^{1/3}$ to get,

$$\frac{200}{3}\cdot\left(\frac{h}{s}\right)-\frac{3{,}400}{3}=0$$

leading to,

$$200h = 3{,}400s$$

Or more conveniently,

$$20h = 340s \qquad (E5)$$

Substituting (E5) into (E3) to get,

$$340s + 170s = 20{,}000$$

This gives a solution for s as,

$$510s = 20{,}000$$

$$\boxed{s = \frac{20{,}000}{510} \approx 39.2157 \qquad (E6)}$$

Similarly substituting (E6) into (E5) and gives a solution for h,

$$20h = 340 \times \frac{20{,}000}{510}$$

$$\boxed{h = \frac{20{,}000}{30} \approx 666.6667 \qquad (E7)}$$

Finally we can evaluate λ by substituting E7 and E6 into E4 as follows,

$$\boxed{\lambda = \frac{20}{3}\cdot\left(\frac{s}{h}\right)^{1/3} = \frac{20}{3}\cdot\left(\frac{30}{510}\right)^{1/3} \approx 2.5927 \qquad (E8)}$$

<u>Step 3: Evaluate the Solution</u>

This means, if we round our results to the nearest integer, that utilizing 667 hours of labour and 39 tonnes of steel gives a maximum revenue of,

$$R(h,s) = 200\,(667)^{2/3}\,(39)^{1/3} \approx \$51{,}777 \qquad \text{(E10)}$$

The Lagrange multiplier λ gives the rate of change of the maximum revenue per unit change in budget. In this case λ = 2.59 indicating that for each additional $1 increase (decrease) in budget would increase (decrease) factory revenue by $2.59.