Exotic Option Pricing using Heston Simulation

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Executive Summary

- Heston Model Calibration is well presented in academic literature such as (Mrazek & Pospisil 2017)
- In these slides we review, derive and present the discretized Heston model simulation process
- This is to allow us to perform Heston simulations for bespoke payoffs and exotic option pricing
- With this presentation we provide the corresponding python implementation

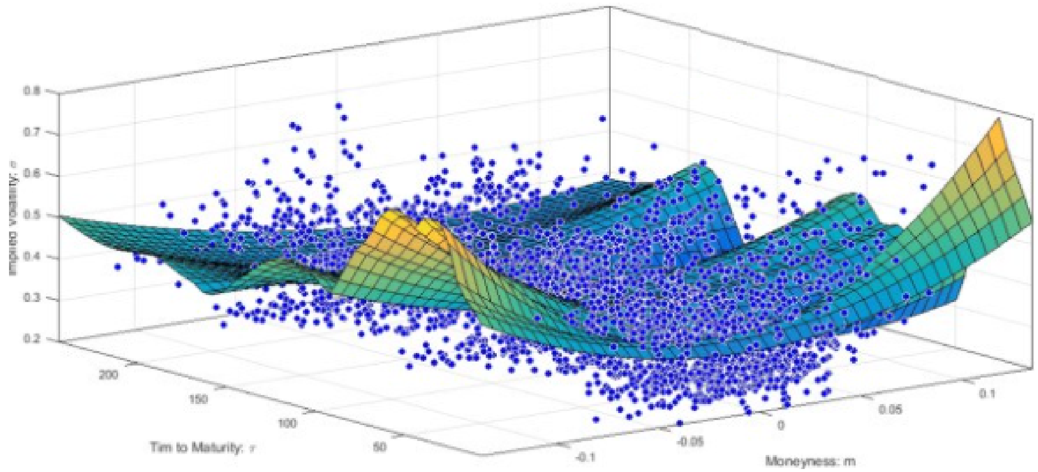
Calibration & Pricing Process

Heston Model Paramters

- Calibrate to Vanilla Options
- Use Closed-Form Methods
- Imply Model Parameters

Exotic Option Pricing

- To Price Complex Payoffs
- Use Heston Simulation
- Given Model Parameters



$$dS(t) = rS(t)dt + s(t)\sqrt{v(t)}dW_S^{\mathbb{Q}}(t)$$
$$dv(t) = \kappa(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_V^{\mathbb{Q}}(t)$$

Heston Model

Heston Stochastic Volatility (SV) Process

For a stock process (S) and a volatility process (v) we have,

$$\begin{aligned} dS(t) &= rS(t)dt + s(t)\sqrt{v(t)}dW_S^{\mathcal{Q}}(t) \\ dv(t) &= \kappa(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_V^{\mathcal{Q}}(t) \end{aligned} \tag{1}$$

with correlated Brownian motions $dW_S^{\mathcal{Q}}(t)dW_V^{\mathcal{Q}}(t) = \rho dt$

Model Parameters

- κ - speed of mean reversion
- $\bar{v}$ - long-term volatility level
- γ - volatility of the volatility

Model Properties

- Recovers implied volatility smile/skew observed in the market
- Especially good for pricing medium and long-dated options[1]
- Stochastic volatility process is mean-reverting and non-negative (CIR Model)
- Market volatilities usually move in opposite direction to underlying asset; the model supports this with negative correlation $\rho_{S,V}$

[1] Heston extensions such as adding jumps helps to better models options with short-dated maturities e.g. Bates model

Log-Normal Process for Better Simulation Convergence

- The Heston model has the following dynamics,

$$\begin{aligned} dS(t) &= rS(t)dt + s(t)\sqrt{v(t)}dW_S^{\mathcal{Q}}(t) \\ dv(t) &= \kappa(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_V^{\mathcal{Q}}(t) \end{aligned} \tag{2}$$

- Defining the log-normal process $X(t) := log(S(t))$ and using Itô's Lemma gives,

$$\begin{aligned} dX(t) &= \left(r - \frac{1}{2}v(t)\right)dt + \sqrt{v(t)}dW_X^{\mathcal{Q}}(t) \\ dv(t) &= \kappa(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}dW_V^{\mathcal{Q}}(t) \end{aligned} \tag{3}$$

with correlation $dW_X^{\mathcal{Q}}(t)dW_V^{\mathcal{Q}}(t) = \rho dt$

Correlated Heston Process

- We use **'Cholesky Decomposition'** to corrlate our Brownian motions, but apply this to the log-normal process $X(t)$.
- This leaves the CIR variance process unmodified and allows exact simulation to be used for the variance process $v(t)$.

$$
\begin{aligned}
dX(t) &= \left(r - \frac{1}{2}v(t)\right) dt + \sqrt{v(t)} \left[\rho_{X,V}\, d\tilde{W}_V^{\mathcal{Q}}(t) + \sqrt{1-\rho_{X,V}^2}\, d\tilde{W}_X^{\mathcal{Q}}(t)\right] \\
dv(t) &= \kappa(\bar{v} - v(t))dt + \gamma\sqrt{v(t)}\, dW_V^{\mathcal{Q}}(t)
\end{aligned} \tag{4}
$$

where $d\tilde{W}^{\mathcal{Q}}$ denotes an independent Brownian motion under $\mathcal{Q}$, the risk-neutral measure

CIR Variance Process

The CIR variance process follows a non-central chi-squared distribution with v(t) conditional on v(s) for $s < t$ as follows,

$$v(t)|v(s) = \bar{c}(t,s) \cdot \chi^2(\delta, \bar{\kappa}(t,s)) \tag{5}$$

where δ represents the degrees of freedom and $\bar{\kappa}$ the critical value of the chi-squared distribution.

The model parameters are: $\bar{c}(t,s) = \frac{\gamma^2}{4\kappa}\left(1 - e^{-\kappa(t-s)}\right)$, $\delta = \frac{4\kappa\bar{v}}{\gamma^2}$ and $\kappa(t,s) = \left(\frac{4\kappa e^{-\kappa(t-s)}}{\gamma^2(1-e^{-\kappa(t-s)})}\right) v(s)$

Exact Simulation of the CIR Process

- Exact simulation allows us perform Monte Carlo simulation with large time steps with no loss of accuracy
- Bypassing **'Euler Discretation'**, requiring small incremental time-steps, gives a significant performance speed-up
- For exact simulation we order the CIR parameters as follows to manage model dependencies,

$$\begin{aligned}
\bar{c}(t_{i+1}-t_i) &= \frac{\gamma^2}{4\kappa}\left(1-e^{-\kappa(t_{i+1}-t_i)}\right) \\
\kappa(t_{i+1}-t_i) &= \left(\frac{4\kappa e^{-\kappa(t_{i+1}-t_i)}}{\gamma^2(1-e^{-\kappa(t_{i+1}-t_i)})}\right)\boxed{v_i} \\
\boxed{v_{i+1}} &= \bar{c}(t_{i+1},t_i)\cdot\chi^2(\delta,\bar{\kappa}(t_{i+1},t_i))
\end{aligned} \tag{6}$$

- For some given initial value $v(t_0)=v_0$ and constant parameter $\delta=4\kappa\bar{v}/\gamma^2$

Almost Exact Simulation of the Heston Process I

Itegrating the Heston processes from (4) leads to,

$$X_{i+1} = X_i + \int_{t+i}^{t_{i+1}} \left(r - \frac{1}{2} v(t) \right) dt + \rho_{X,V} \boxed{\int_{t+i}^{t_{i+1}} \sqrt{v(t)}\, d\tilde{W}_V^{\mathcal{Q}}(t)} + \sqrt{1 - \rho_{X,V}^2} \int_{t+i}^{t_{i+1}} \sqrt{v(t)}\, d\tilde{W}_X^{\mathcal{Q}}(t) \tag{7}$$

and

$$v_{i+1} = v_i + \kappa \int_{t+i}^{t_{i+1}} (\bar{v} - v(t)) dt + \gamma \boxed{\int_{t+i}^{t_{i+1}} \sqrt{v(t)}\, dW_V^{\mathcal{Q}}(t)} \tag{8}$$

Heston Variance Integral

The variance in the Heston asset process $X(t)$ from (7) is the same as that in the CIR variance process (8), which we rearrange to give,

$$\boxed{\int_{t+i}^{t_{i+1}} \sqrt{v(t)}\, dW_V^{\mathcal{Q}}(t)} = \frac{1}{\gamma}\left(v_{i+1} - v_i - \kappa \int_{t+i}^{t_{i+1}} (\bar{v} - v(t))dt \right) \tag{9}$$

We can simulate the variance v_{i+1} for a given v_i using the CIR process dynamics or via the CIR sequence (6), which employs the non-central chi-squared distribution.

Almost Exact Simulation of the Heston Process II

Simulating the Heston asset process $X(t)$ from (7) and applying the variance integral definition (9) gives,

$$
\begin{aligned}
X_{i+1} = X_i &+ \int_{t+i}^{t_{i+1}} \left(r - \frac{1}{2} v(t) \right) dt \\
&+ \frac{\rho_{X,V}}{\gamma} \left(v_{i+1} - v_i - \kappa \int_{t+i}^{t_{i+1}} (\bar{v} - v(t)) dt \right) \\
&+ \sqrt{1 - \rho_{X,V}^2} \int_{t+i}^{t_{i+1}} \sqrt{v(t)}\, d\tilde{W}_X^{\mathcal{Q}}(t)
\end{aligned} \tag{10}
$$

Integral Approximation

- Evaluating the integrals in (10) numerically is computationally expensive requiring function evaluation at many time-steps
- Therefore we evaluate the integrals using a 'freezing' approximation, which is consistent with Euler discretisation schemes, where left-integration boundaries are used to perform piecewise-constant integration
- Consequently we cannot perform exact simulation with large time-steps without losing accuracy. However with a moderate number of time-steps this approximation gives good results
- Performance is greatly improved with negligible approximation error

Heston Model Simulation Process I

Applying the freezing approximation to the Heston model simulation of $X(t)$ from (10) leads to the following, where $v(t)$ terms become v_i,

$$
\begin{aligned}
X_{i+1} \approx X_i &+ \int_{t+i}^{t_{i+1}} \left(r - \frac{1}{2} \boxed{v_i} \right) dt \\
&+ \frac{\rho_{X,V}}{\gamma} \left(v_{i+1} - v_i - \kappa \int_{t+i}^{t_{i+1}} (\bar{v} - \boxed{v_i}) dt \right) \\
&+ \sqrt{1 - \rho_{X,V}^2} \int_{t+i}^{t_{i+1}} \sqrt{\boxed{v_i}}\, d\tilde{W}_X^{\mathbb{Q}}(t)
\end{aligned}
\tag{11}
$$

Heston Model Simulation Process II

This leads to a trivial discretization for X_{i+1} as follows,

$$\begin{aligned} X_{i+1} \approx X_i &+ \left(r - \frac{1}{2} v_i \right) \Delta t \\ &+ \frac{\rho_{X,V}}{\gamma} \left(v_{i+1} - v_i - \kappa(\bar{v} - v_i) \Delta t \right) \\ &+ \sqrt{\left(1 - \rho_{X,V}^2\right) v_i} \left(\tilde{W}_X^{\mathcal{Q}}(t_{i+1}) - \tilde{W}_X^{\mathcal{Q}}(t_i) \right) \end{aligned} \tag{12}$$

where $\tilde{W}_X^{\mathcal{Q}}(t_{i+1}) - \tilde{W}_X^{\mathcal{Q}}(t_i) \stackrel{d}{=} \sqrt{\Delta t} Z_X$ with $Z_X \sim \mathcal{N}(0, 1)$

Discretization of Almost Exact Heston Model

To implement the Almost Exact Simulation (AES) of the Heston model, we discretize the Heston model as follows,

$$\begin{aligned} X_{i+1} &\approx X_i + k_0 + k_1 v_i + k_2 v_{i+1} + \sqrt{k_3 v_i} Z_X \\ v_{i+1} &= \bar{c}(t_{i+1}, t_i) \cdot \chi^2(\delta, \bar{\kappa}(t_{i+1}, t_i)) \end{aligned} \tag{13}$$

with constants $k_0 = \left(r - \frac{1}{2} v_i\right) \Delta t$, $k_1 = \left(\frac{\rho_{X,V}}{\gamma} - \frac{1}{2}\right) \Delta t - \frac{\rho_{X,V}}{\gamma}$, $k_2 = \frac{\rho_{X,V}}{\gamma}$ and $k_3 = \left(1 - \rho_{X,V}^2\right) \Delta t$,

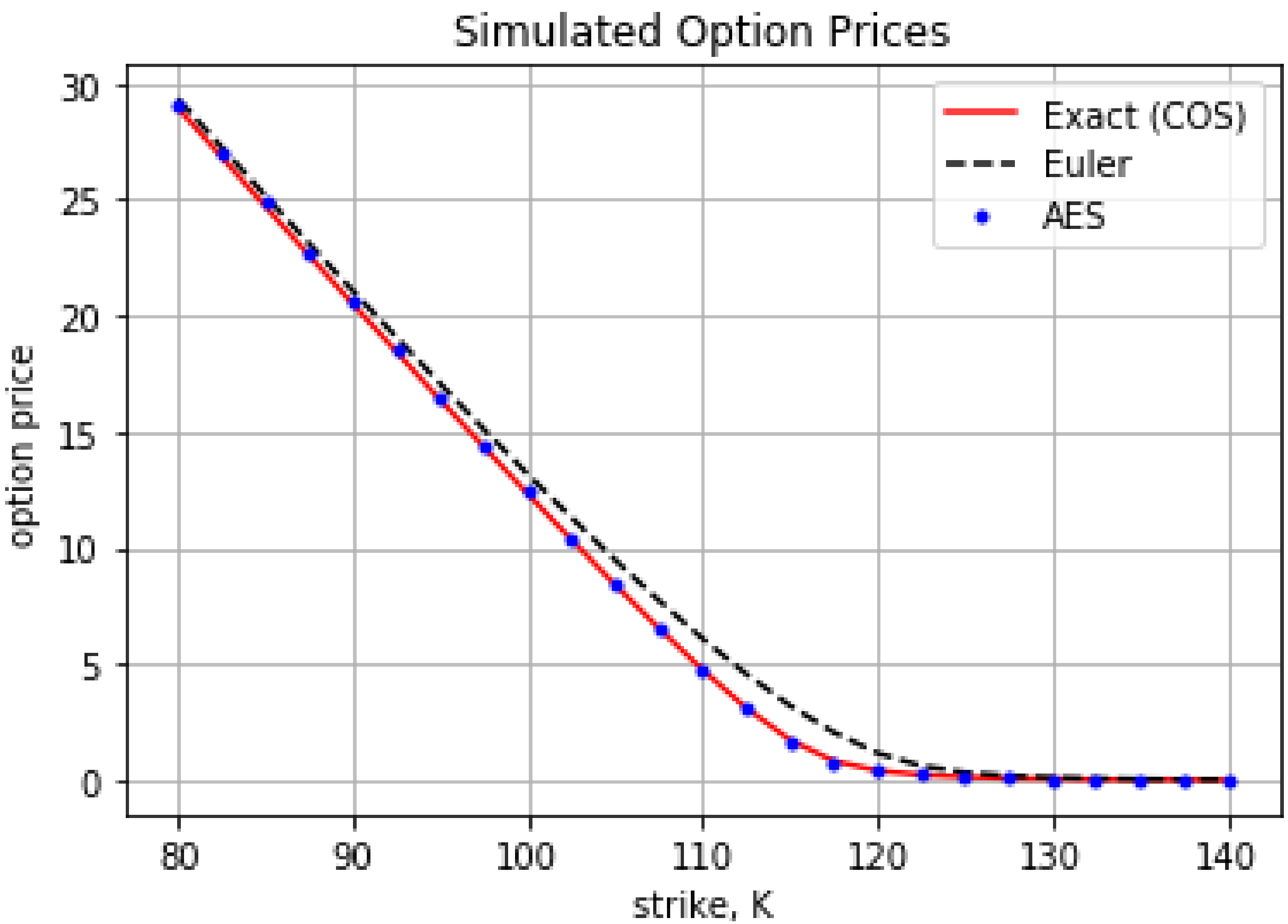
and the variance process simulated as: $\bar{c} = \frac{\gamma^2}{4\kappa}\left(1 - e^{-\kappa(t_{i+1} - t_i)}\right)$, $\delta = \frac{4\kappa\bar{v}}{\gamma^2}$, $\bar{\kappa} = \left(\frac{4\kappa e^{-\kappa\Delta t}}{\gamma^2(1 - e^{-\kappa\Delta t})}\right) v_i$ and $\chi^2(\delta, \bar{\kappa})$ the non-central chi-squared distribution with δ degrees of freedom and non-centrality parameter $\bar{\kappa}$.

Simulation Remarks

- When simulating the CIR process variance cannot be negative
- Negative variance is avoided when the Feller condition is satisfied i.e. when $2\kappa\bar{v} > \gamma^2$
- This is often not the case, so we manage this problem using the 'absorption' technique where $v_{i+1} = v_{i+1}^{+}$ or equivalently $v_{i+1} = max(v_{i+1}, 0)$
- We could also use the 'reflection' technique with $v_{i+1} = |v_{i+1}|$
- Empirically the absorption technique exhibits lower bias

Benchmark Pricing Results I

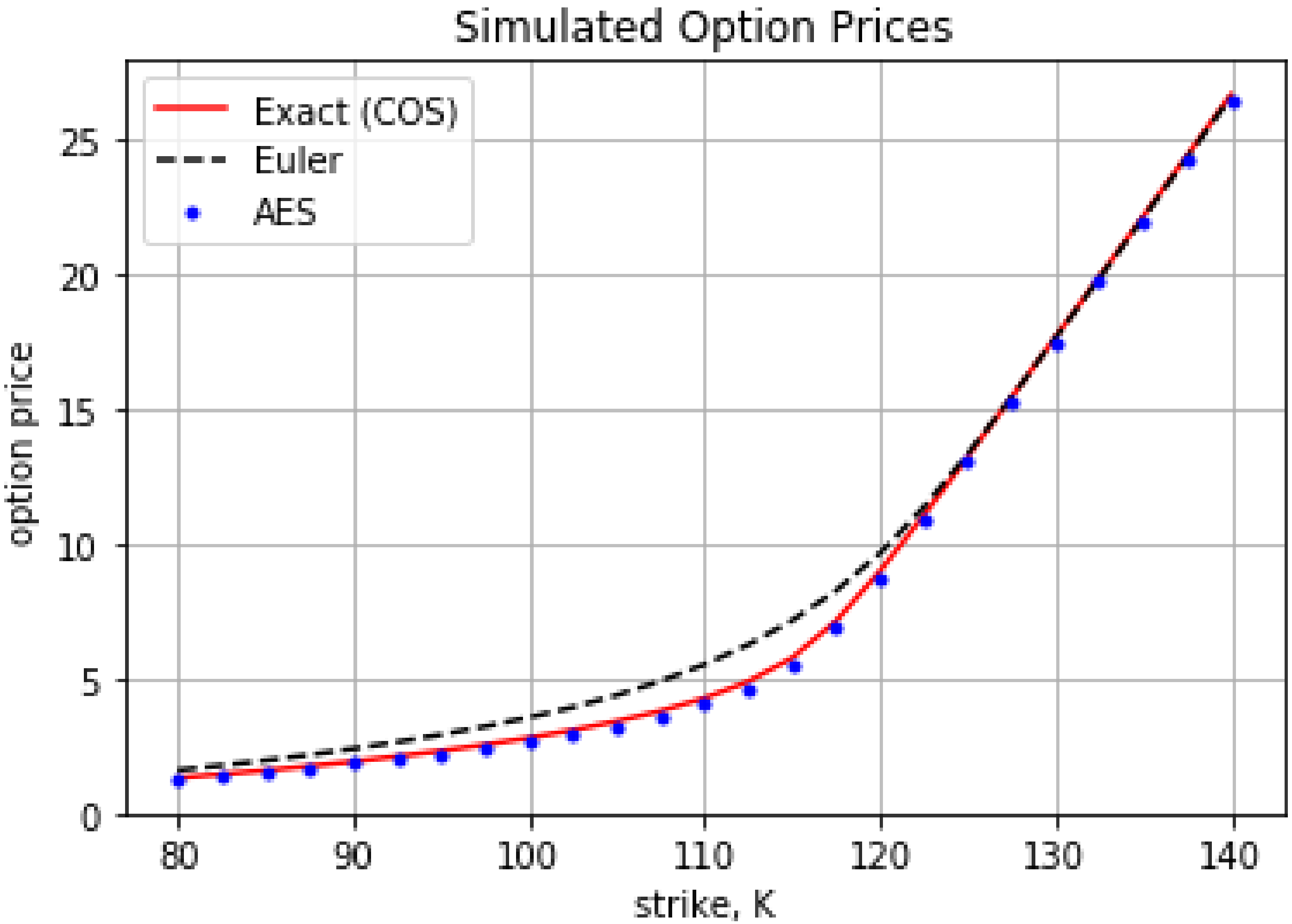
European Call Option[2] Convergence



[2]Note: We use Strike on the x-axis not Spot

Benchmark Pricing Results II

European Put Option[3] Convergence



[3] Note: We use Strike on the x-axis not Spot

Simulation Convergence Results

Standard Error by Timestep Size

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Euler Scheme
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Strike (K), Timestep (dt), Standard Error (eps)
Euler Scheme, K = [140], dt = 1.0, eps = [1.30476871]
Euler Scheme, K = [140], dt = 0.25, eps = [0.54404403]
Euler Scheme, K = [140], dt = 0.125, eps = [0.1720427]
Euler Scheme, K = [140], dt = 0.0625, eps = [0.08078707]
Euler Scheme, K = [140], dt = 0.03125, eps = [0.01260589]
Euler Scheme, K = [140], dt = 0.015625, eps = [0.00605332]

Almost Exact Simulation (AES)
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Strike (K), Timestep (dt), Standard Error (eps)
AES Scheme, K = [140], dt = 1.0, eps = [0.00800533]
AES Scheme, K = [140], dt = 0.25, eps = [0.00985109]
AES Scheme, K = [140], dt = 0.125, eps = [0.00135139]
AES Scheme, K = [140], dt = 0.0625, eps = [0.00661993]
AES Scheme, K = [140], dt = 0.03125, eps = [0.0157029]
AES Scheme, K = [140], dt = 0.015625, eps = [0.00699352]
```

Final Remarks

Almost Exact Simulation (AES)

- Easily replicates exact option prices
- Integral freezing approx has negligible impact on price
- AES simulation is quick and requires few simulation time-steps

Euler Discretization Scheme

- Euler performs well, but slower & requires more time steps

Exotic Options & Bespoke Option Payoffs

- Knowing the Heston simulation & descretization process from (13) all that remains is to script the bespoke or exotic payoff, which is a trivial exercise.

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