

Computationally Efficient Zero Coupon Swap Formulae

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In this short paper, we outline computationally efficient formulae for zero coupon swap pricing.

Zero coupon rates are defined as,

$$\text{Zero Coupon Rate, } Z_n = \prod_{i=1}^n (1 + r_i \tau_i) - 1 \quad (1)$$

We note that discount factors can be calculated using **simple compounding** as below,

$$DF(t_n) = \frac{1}{(1 + r_1 \tau_1)(1 + r_2 \tau_2) \dots (1 + r_n \tau_n)} = \frac{1}{\prod_{i=1}^n (1 + r_i \tau_i)} \quad (2)$$

An alternative formula for the zero coupon rate that is **computationally efficient** can be derived by substituting (2) into (1),

$$\text{Zero Coupon Rate, } Z_n = \left(\frac{1}{DF_n} \right) - 1 \quad (3)$$

Therefore¹ the PV of a zero coupon swap leg can be calculated as follows,

$$\begin{aligned} PV(\text{Zero Coupon Leg}) &= N Z_n DF_n \\ &= N \left[\prod_{i=1}^n (1 + r_i \tau_i) - 1 \right] DF_n \\ &= N \left[\left(\frac{1}{DF_n} \right) - 1 \right] DF_n \\ &= N (1 - DF_n) \end{aligned} \quad (4)$$

where N denotes the trade notional, Z the zero coupon rate, r_i the fixing rate (or reset), τ_i the fixing year fraction and DF the discount factor.

¹ Typically for swaps $PV(\text{Coupon}) = N r \tau DF$, but here the zero rate Z_n explicitly incorporates the fixing year fraction term, τ .

For zero coupon rates **with historical fixings** equation (1) becomes,

$$\text{Zero Coupon Rate, } Z_n = \underbrace{\prod_{\{i < t\}} (1 + r_i \tau_i)}_{\text{Accrued}} \underbrace{\prod_{\{i \geq t\}} (1 + r_i \tau_i)}_{\text{Projected}} - 1 \quad (5)$$

where t denotes today, $\{i < t\}$ the set of known historical fixings and $\{i \geq t\}$ the projected fixings in the future.

Likewise the **computationally efficient** zero rate **with historical fixings** from equation (3) becomes,

$$\text{Zero Coupon Rate, } Z_n = \underbrace{\prod_{\{i < t\}} (1 + r_i \tau_i)}_{\text{Fixings}} \left(\frac{1}{DF_n} \right) - 1 \quad (6)$$

Finally the PV of a zero coupon swap leg **with historical fixings** can be calculated as,

$$\begin{aligned} PV(\text{Zero Coupon Leg}) &= N Z_n DF_n \\ &= N \left[\underbrace{\prod_{\{i < t\}} (1 + r_i \tau_i)}_{\text{Accrued}} \underbrace{\prod_{\{i \geq t\}} (1 + r_i \tau_i)}_{\text{Projected}} - 1 \right] DF_n \\ &= N \left[\prod_{\{i < t\}} (1 + r_i \tau_i) \left(\frac{1}{DF_n} \right) - 1 \right] DF_n \\ &= N \left(\underbrace{\prod_{\{i < t\}} (1 + r_i \tau_i)}_{\text{Fixings}} - DF_n \right) \end{aligned} \quad (7)$$