OTS-03 Summary Document

Overview

This document summarises the OTS-03 lecture on option pricing models, volatility measurement and forecasting. The lecture discusses the features of options, their relation with the underlying asset, put-call parity, risk neutrality and the Black-Scholes-Merton model for option pricing. In addition to this, this lecture also covers the methods for measurement and forecast of volatility. After finishing the lecture, you should be able to understand the underlying principles of option pricing and volatility measurement along with its forecast.

The following topics are covered:

- Model-independent features of options
- Put-Call Parity
- Option pricing variables and parameters
- A toy binomial pricing model
- Risk-neutrality
- Deriving the Black-Scholes-Merton PDE
- Properties of BSM solution
- Option Greeks
- Volatility measurement
- Volatility forecasting

Model Independent features of options

- **American > European**: If you don't choose to exercise, an American option is a European option. So the American option is European plus some extra value.

- **Longer Dated American Options > Short Dated**: You can turn the longer-dated option into the short one by exercising the longer-dated American option. So it is a short-dated option plus some more optionality.

- **Call < Underlying**: If you exercise, you get the underlying asset. So if Call price > stock price, sell the call and buy the underlying asset.

- **Put < Strike**: If you exercise, you get the strike minus the current price. So the highest value the option can ever have is the strike (if S=0).

- **Lower Strike C > Higher Strike C**: There will be times when a lower strike call can be exercised, and the higher strike call cannot.

If these model-independent features do not adhere in the real market, there will be an arbitrage opportunity.

Put/Call Parity

- Put-call parity defines the relationship between the puts and calls. It is important for put-call parity to hold as there will be an arbitrage opportunity otherwise.
- The relationship is given as -

$$S + P = C + X e^{-rT}$$

- This equation shows that buying a stock and a put is equivalent to buying a call and a zero-coupon bond with the future value equivalent to the strike price.

The following snapshot shows the initial value and value at expiration for multiple instruments.

Instrument	Initial Value	Value at Expiration	
		If S<X	If S> X
Long P(X)	P(X)	(X-S)	0
Short C(X)	-C(X)	0	-(S- X)
Long Stock	S	S	S
Total Portfolio	S+ P(X) -C(X)	X	X

Note

- A call is like borrowing money to buy stock and a put, i.e. it is a limited downside leveraged position in the underlying.
- The payoff from any instrument can be replicated with the other three.
- For call and put to have equal value, the strike price must be the forward price.
- The relationship does not hold for American options as we can't be sure of the duration of a short option position.

- If we can make a bullish options position into a bearish options position by adding stock, then we should also be able to neutralize an option directionally.
- So P/C parity is an important precursor to volatility trading.

Option Pricing: Inputs

- Underlying price.
- Strike: Distance between strike and price define the option.
- Interest rates: Rates are the discount factor for cash flows.
- Dividends: Income from the stock which affects its future value.
- Time.
- Volatility.

A toy model: one period binomial

- We have a $100 stock that in the next period can go up to $150, or down to $90. What is the 100 call worth?
- We sell the call and hedge the directional risk by buying 'h' shares (we don't know what 'h' is yet).
- So portfolio value is $S\ x\ h\ -\ C$, where S is the stock price and C is the call option price.
- At $150, our portfolio is worth $150 x h - $50. $50 is the call option value as when the price is $150, the underlying moved $50 (from 100 to 150).
- At $90, our portfolio is worth $90 x h as the call is out-of-money, so C=0.
- For us to be hedged, we need to be indifferent to these states so
  $90 x h =$150 x h - $50 or h=5/6.
- At $150, our portfolio is worth $150 x (5/6) - $50 = $75
- At $90, our portfolio is worth $90 x (5/6) = $75
- So initial fair value, Sxh - C = final value
- (5/6) x $100 – C=$75 or C=$8.33

Risk neutrality

Generally, people are risk-averse. This means they will accept a smaller certain amount over a risky bet with a higher average payout. A risk-neutral person would be indifferent. All they want to do is to maximize the expected value. In our case, we have shown that the option is valued as if there was no difference between the hedge and the uncertain payoff of the option.

Risk neutrality: An example

- **A forward contract:** an agreement to buy or sell something at a set time in the future.
- This is priced to avoid arbitrage now and only involves interest and carry rates.
- Ideas about asset appreciation are irrelevant but clearly, both parties will have a view on this.

Black Scholes Model

The Black Scholes model provides the analytical framework for valuing options. It is the most commonly used option pricing model. It takes into account the strike price, the stock price, time to expiry, volatility and the interest rate and gives the option value as an output.

We can create a hedged portfolio by selling h shares short.

$$Portfolio \; = \; C \; - hS$$

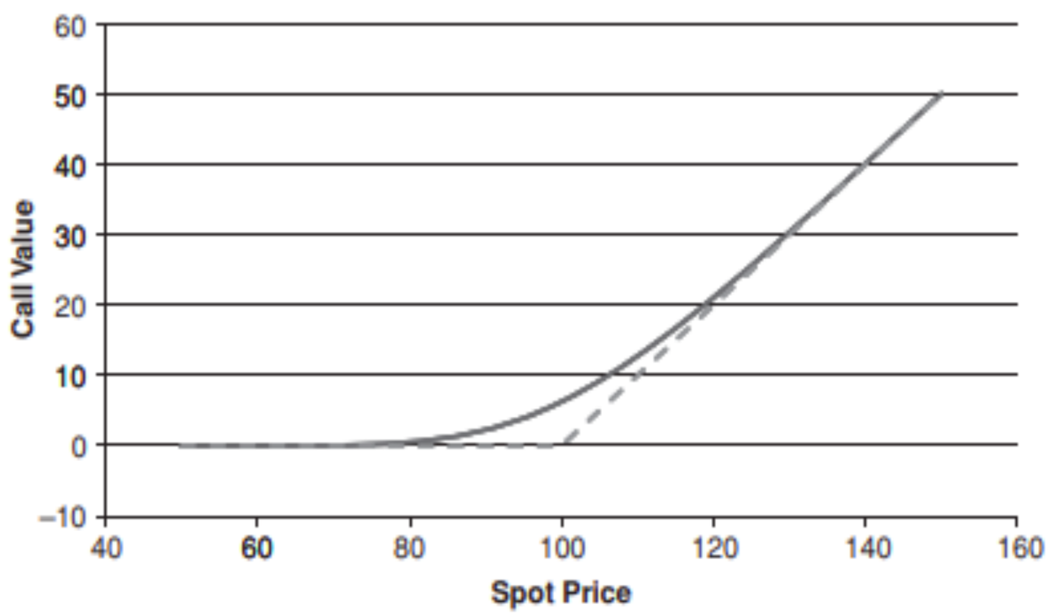
The BSM differential equation is given by:

$$1/2\, \sigma^2 S^2 \Gamma^{\blacksquare} \; + \; \theta \; = \; 0$$

Solution for European Options

The option price is essentially an interpolation between the stock price and the bond price.

The following snapshot shows the BSM solutions for European options for a 100 strike call



Option value

- The option value is given by -

$$C = SN(d_1) - Xexp(-rT)N(d_2)$$

- N() is the cumulative normal distribution.

$$d_1 = \frac{ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)\mathrm{T}}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where,

S■ is the underlying stock price,
X is the strike price,
r - risk-free interest rate,
T - time to expiration,
σ - Standard deviation of log returns.

- A call is a part ($N(d_1)$) of the stock and a part ($-e^{-rT}N(d_2)$) of the cash/bond.
- $N(d_1)$is the amount of stock we are exposed to.
- $N\ (d_2)$is the amount of cash we are exposed to, i.e. what we need to deliver at the expiration
- This interpretation is only true in a risk-neutral world; a world that doesn't exist.
- In the real world, the probabilities are mainly driven by drift, something we have no interest in (as a pricing variable) or significant ability to predict.

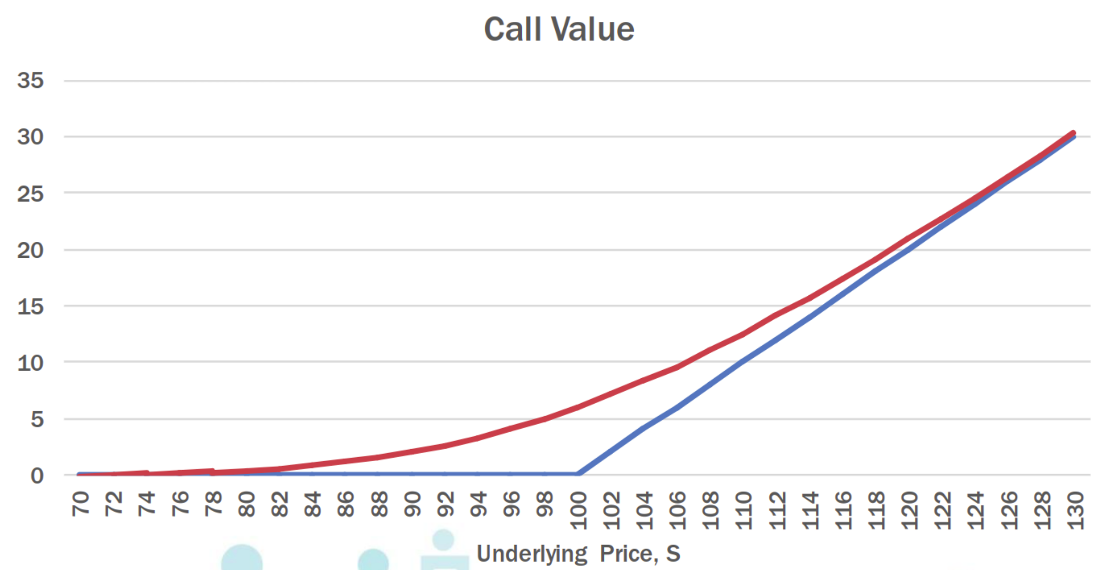
Greeks

- "Greeks" is the collective term for partial derivatives (in the mathematical sense) of the option price. Volatility traders generally quantify their positions in terms of greek exposure. Since the BSM equation is linear, greeks can be aggregated across a position by adding the greeks of each option.

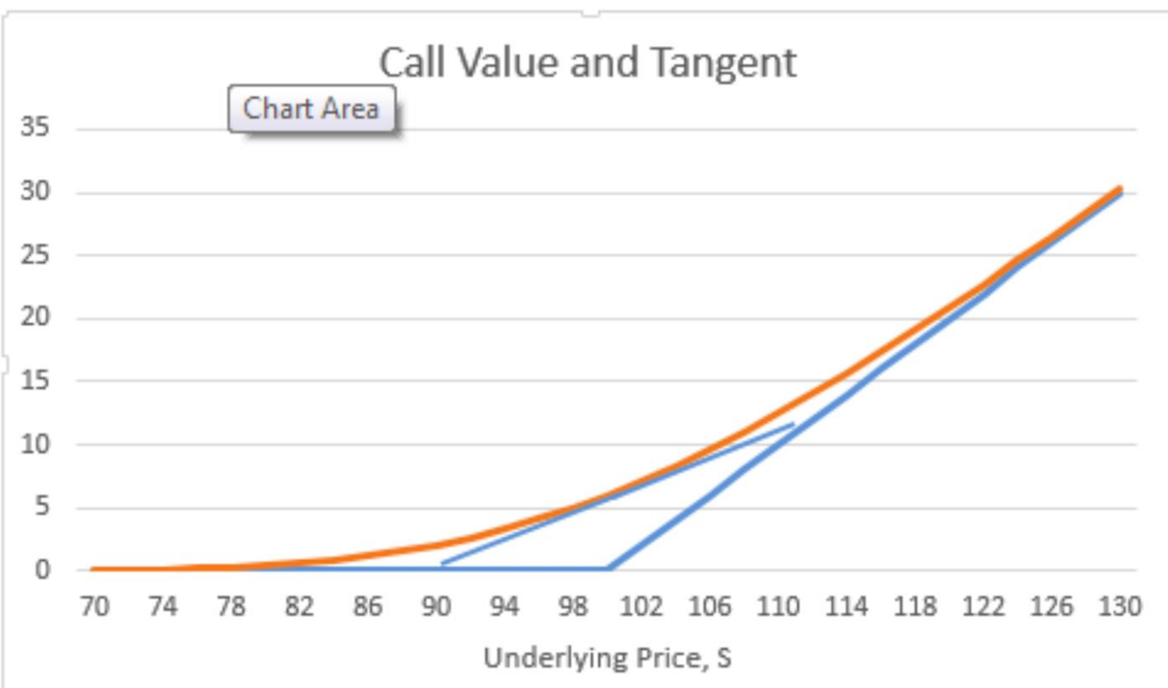
Delta

- Delta is the partial derivative of the option price w.r.t. underlying.
- Delta is the tangent of the option price vs the underlying graph

The following snapshot shows the expiration value (blue) and current premium (red) for a call with a strike price of 100

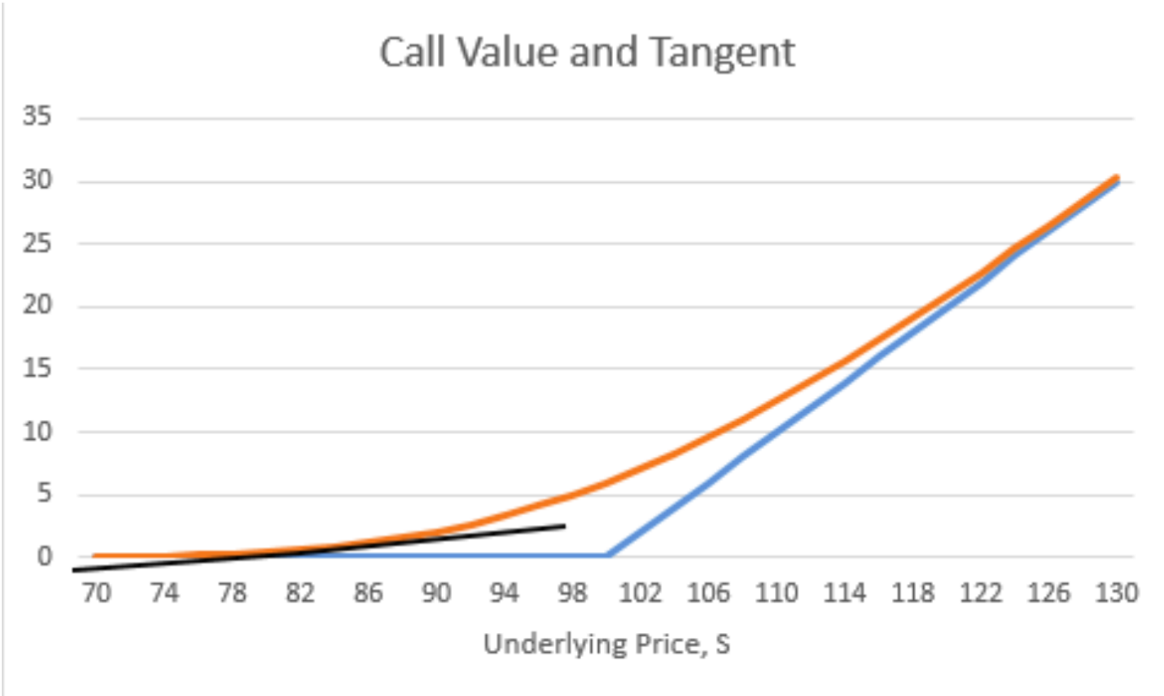


The following snapshot shows the tangent of a call value.



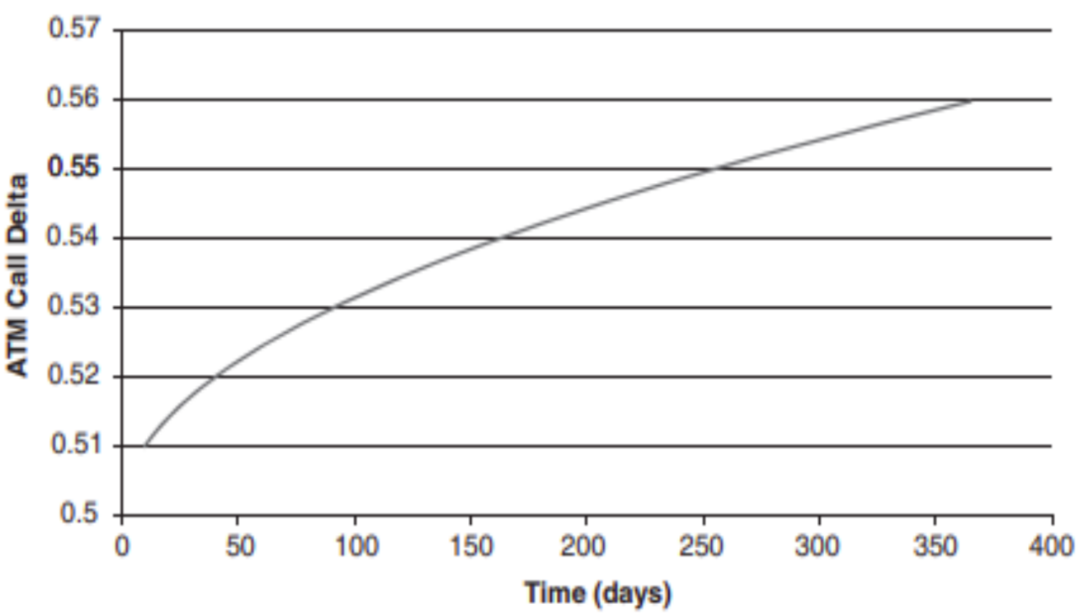
- If the underlying price is much lower than the strike, the option is way out-of-the-money, and the delta will be almost zero.

The following snapshot shows the delta line when S=80



Delta vs time

The following snapshot shows the ATM call delta value over time.



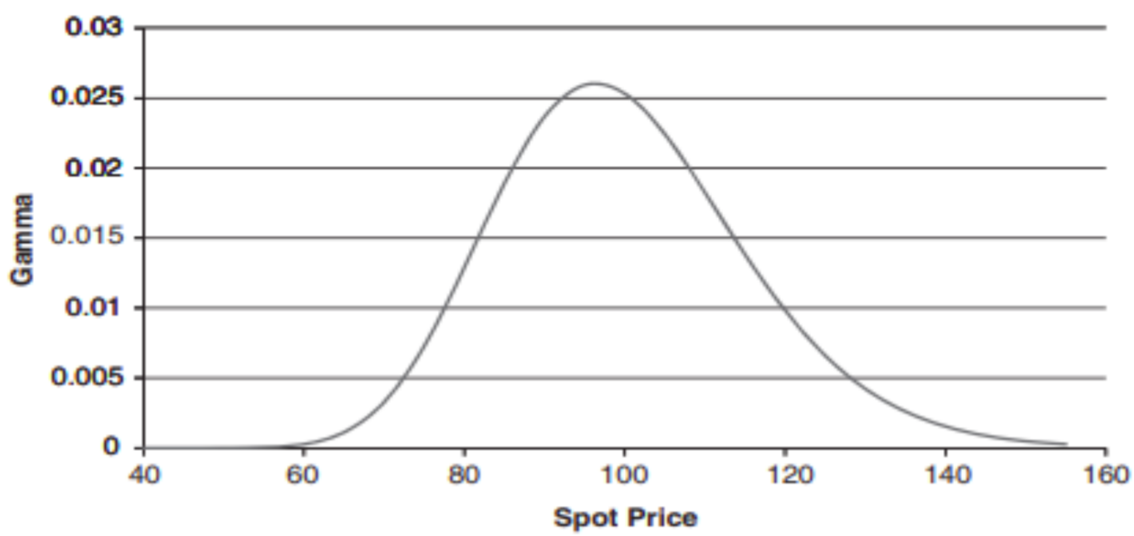
- At the infinite time (or volatility) call option delta goes to one and put option delta to zero.
- At high volatility, the stock can go much higher but is bounded below as it can't go below 0.

Gamma

- Gamma is the partial derivative of the delta w.r.t. the underlying. Simply put, gamma is the rate of change of delta. To manage delta, you must know gamma. Gamma is highest for ATM short-dated options.

Gamma vs Spot

The following snapshot shows the plot with spot price on the x-axis and the respective gamma values on the y-axis for a 100 strike call.

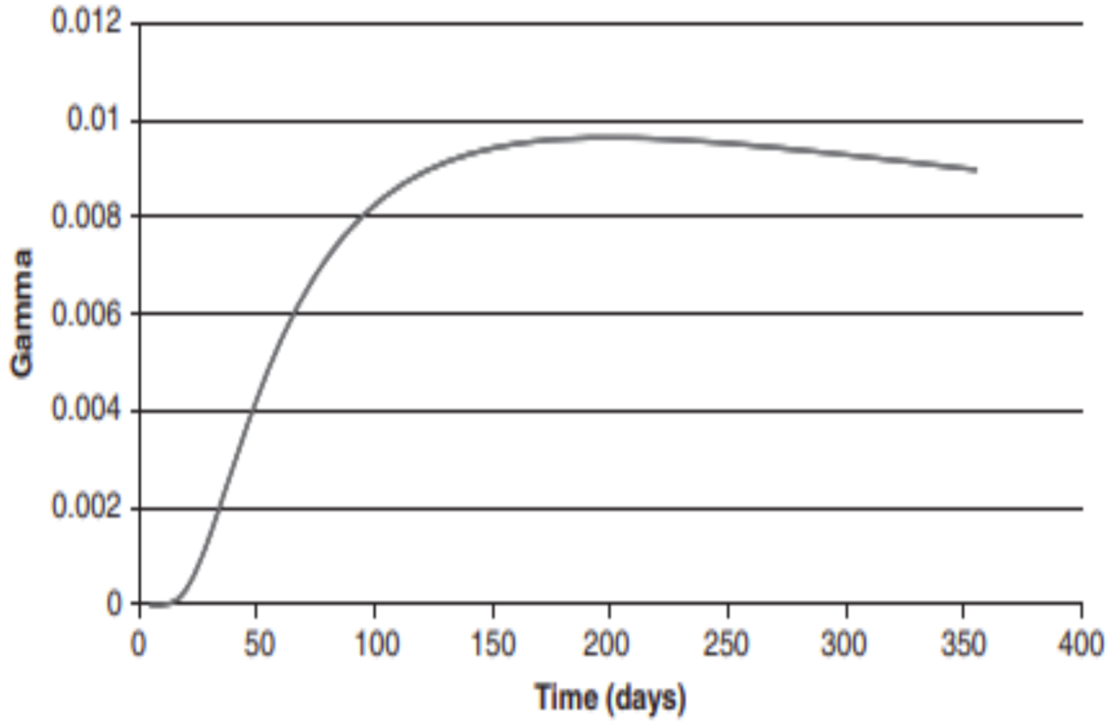


- Unsurprisingly, delta looks a lot like a cumulative normal distribution, and gamma looks a lot like a normal distribution.

Gamma vs Time

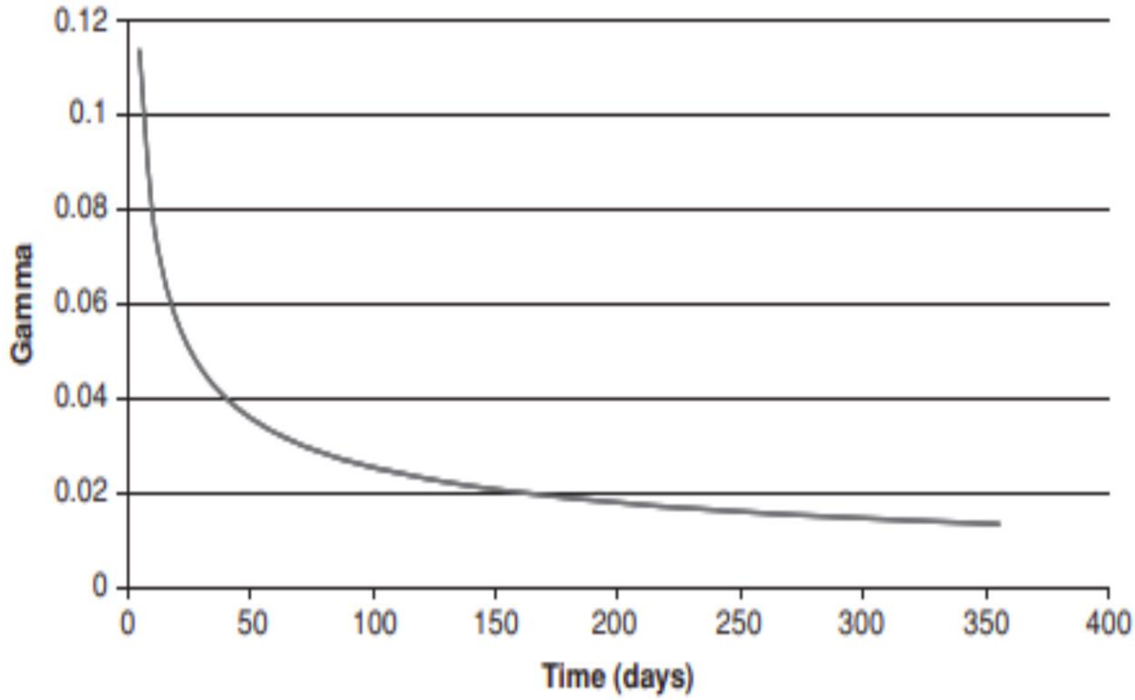
Out-of-the-money option

The following snapshot shows the gamma vs time plot of out-of-the-money options.

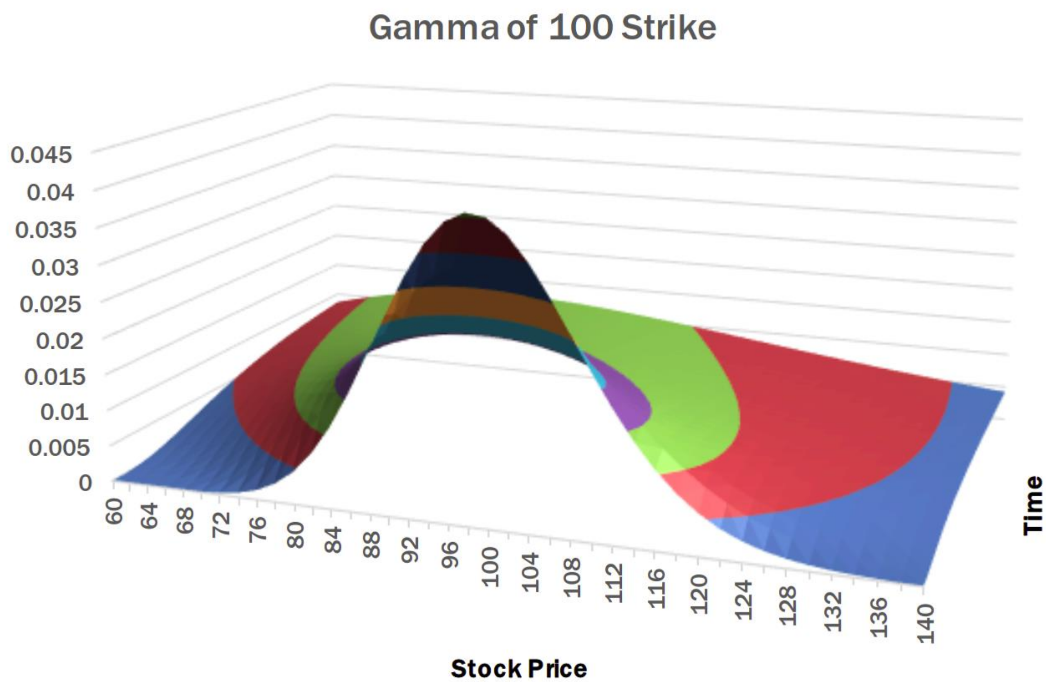


At-the-money option

The following snapshot shows the gamma vs time plot of at-the-money options.



The following snapshot shows the 3D plot with the price on the x-axis, gamma on the y-axis and time on the z-axis.



Theta

- Theta is the partial derivative of the option price w.r.t. time.
- The formula is given by -

$$\theta_{call} = -\frac{S\sigma n(d_1)}{2\sqrt{t}} - rXexp(-rT)N(d_2)$$
$$\theta_{put} = -\frac{S\sigma n(d_1)}{2\sqrt{t}} + rXexp(-rT)N(-d_2)$$

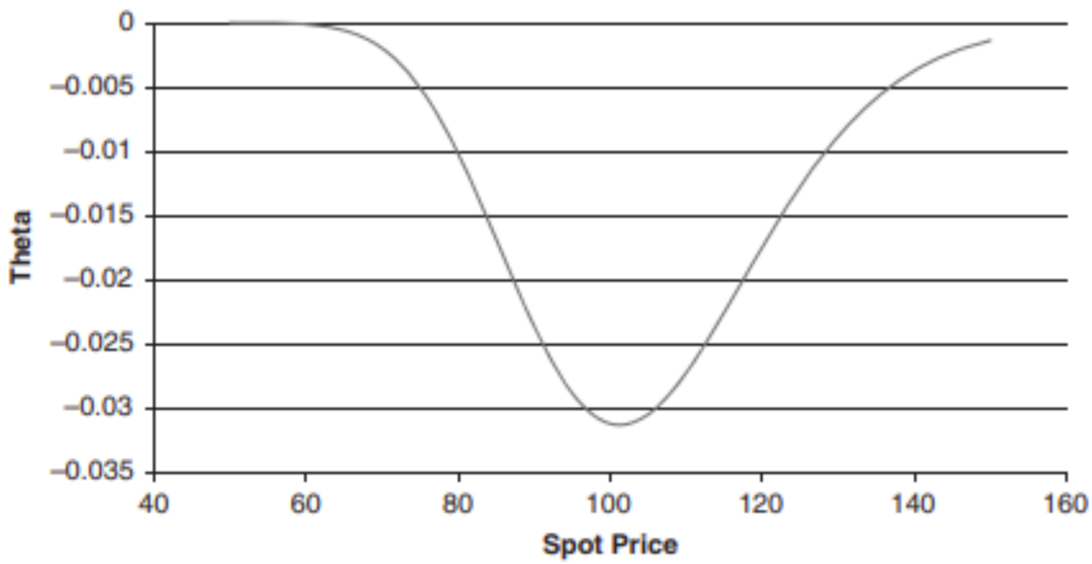
- The first term (same for puts and calls) is the time decay, the amount the option declines because volatility has less time to act.
- The second term is often misunderstood (or just ignored).

$$\theta_{call} - \theta_{put} = rXexp(-rT)$$

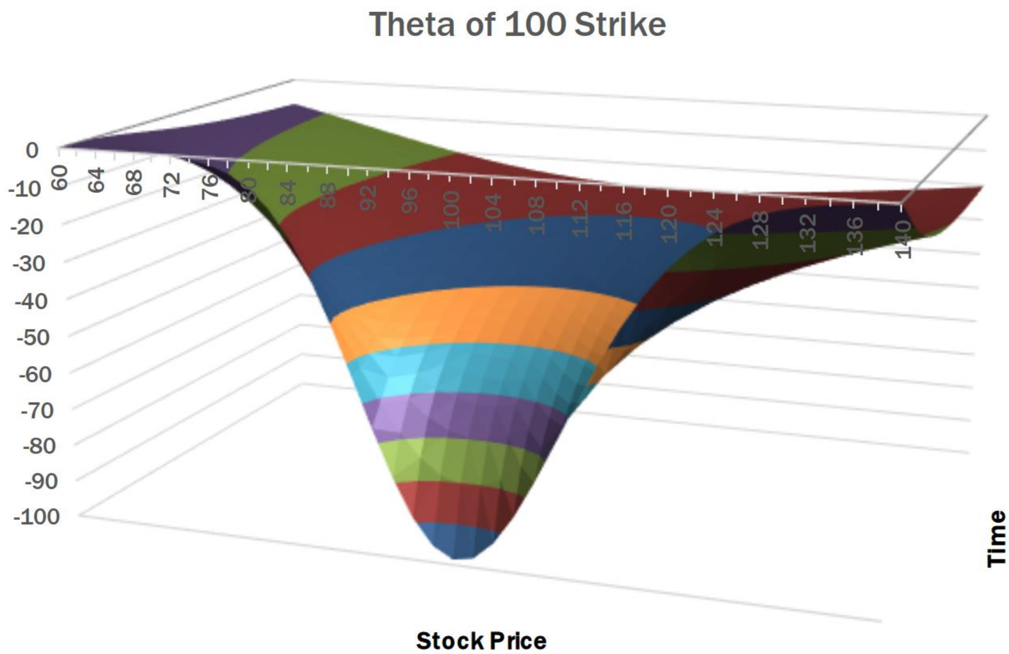
- From the Put-call parity discussion, we know a long call and short put has no volatility exposure. However, it does have to carry the cost of holding the strike value in cash.
- Theta allocates this carry across calls and puts according to the chance of each finishing in the money.

Theta vs Spot

The following snapshot shows the plot with spot price on the x-axis and the respective theta values on the y-axis for a 100 strike call.



The following snapshot shows the 3D plot with the price on the x-axis, theta on the y-axis and time on the z-axis.

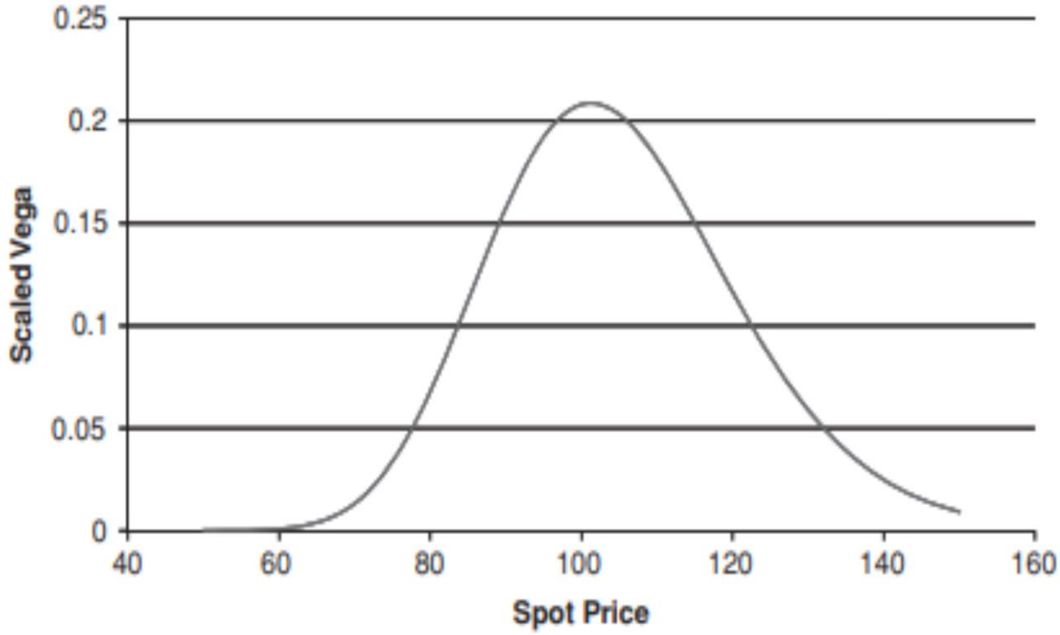


Vega

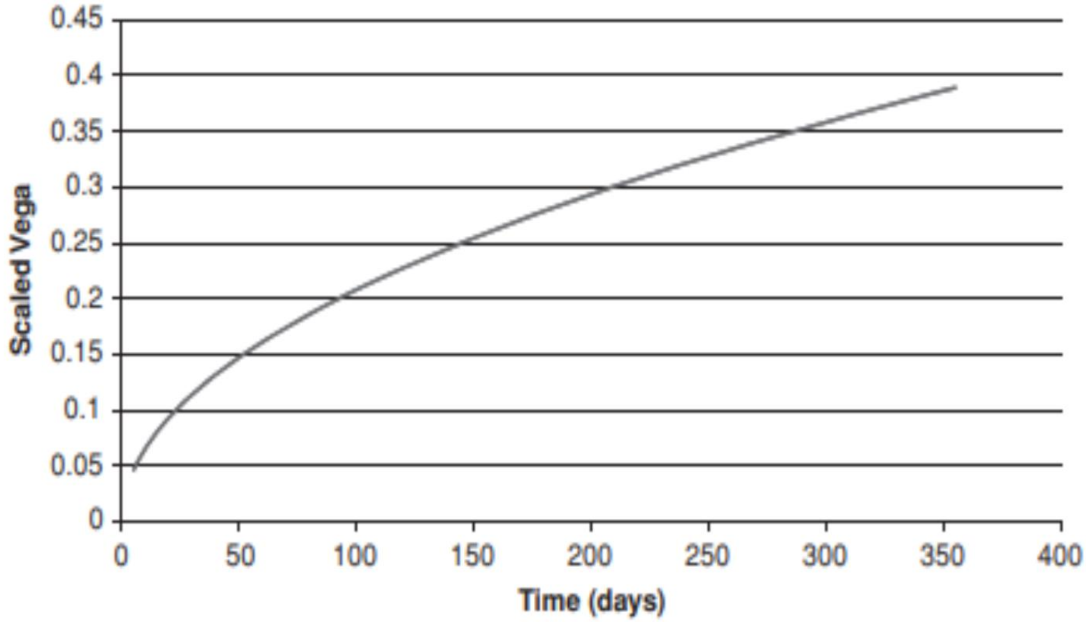
- Vega is the partial derivative of the option price w.r.t. volatility.
- It's inconsistent with the assumption of constant volatility but very important to traders.

$$Vega_{call} = Sn(d_1)\sqrt{t} = Vega_{put}$$

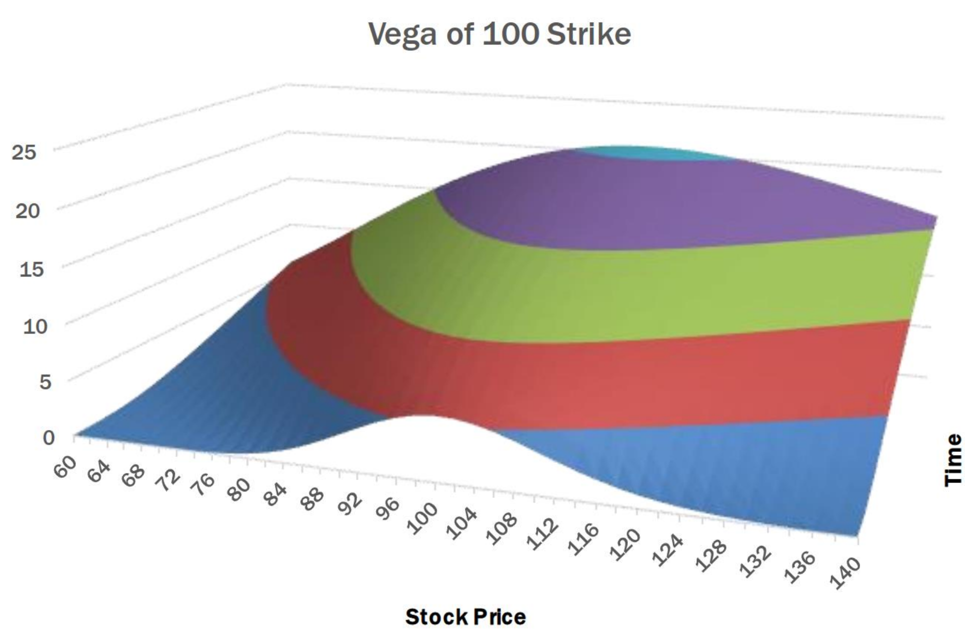
The following snapshot shows the plot with spot price on the x-axis and the respective vega values on the y-axis for a 100 strike call.



The following snapshot shows the plot with time on the x-axis and the respective vega values on the y-axis.



The following snapshot shows the 3D plot with the price on the x-axis, vega on the y-axis and time on the z-axis.



Minor greeks: Rho

- Rho is the partial derivative of the option price w.r.t. interest rates.
- It's inconsistent with the assumption of constant rates.
- Usually ignored by traders, as it is managed at the firm level by the treasury.
- Generally scaled to be dollar change for a 1% move in rates.

$$\rho_c = TXexp(-rT)N(d_2)$$

$$\rho_p = -TXexp(-rT)N(-d_2)$$

Secondary greeks

- From just looking at equations or graphs it is clear that the Greeks also have derivatives, e.g. the derivative of delta w.r.t. volatility, the derivative of Vega with respect to time, etc.
- Any professional-level trading or risk system will show these greeks, but a detailed understanding of their characteristics isn't essential.
- There is no standard nomenclature for these derivatives, e.g. the derivative of the delta with respect to volatility is variously called "Vanna", "DdeltaDvol", "DdelV" or even "alpha", depending on who the trader learned from.

__Trading screen__

The following snapshot shows the trading screen with bid-ask quotes for options.

What is volatility?

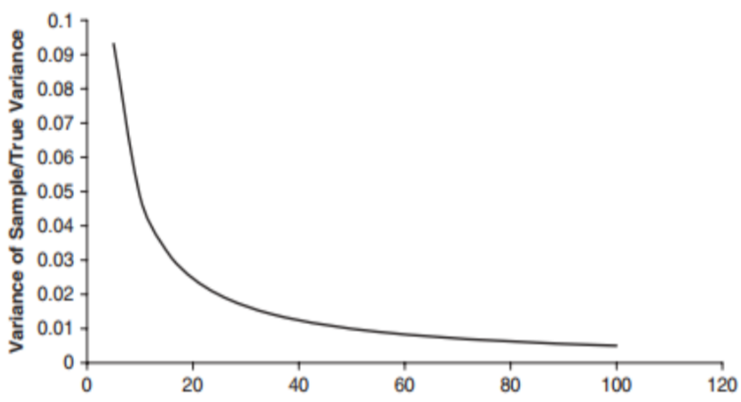
Volatility is the amount and frequency of price changes. It represents the uncertainty in the market and is defined as the standard deviation of the log returns. Variance is given by -

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}\left(r_i - \bar{r}\right)^2$$

$$r_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

The challenge with this is about the value of N. The error in true volatility as compared to the one calculated using the sample data is called the sampling error. It decreases as N increases.

The following snapshot shows the error decreases as N increases.



Therefore, people look for other methods for estimating population volatility using sample volatility. For example -

Parkinson Estimator

- Uses daily range with Day's High and Low

$$\sigma = \sqrt{\frac{1}{4N \ln 2} \sum_{i=1}^{N} \left(\ln \frac{h_i}{l_i} \right)^2}$$

- Where *N* is the number of days of lookback data used, h and *l* are the High and Low of the day for the lookback period.
- This estimator has greater precision.
- Generally, it is used with 30 days of data.
- It has a drawback that it considers only movements during market hours. It doesn't consider the aftermarket hours movements.

Other estimators that can be used are Garman Klass, Rogers-Satchel, Yang-Zhang.

Volatility estimation and forecasting

The following are the reasons for forecasting volatility:

- To check for stable and less risky stocks. If we know how to calculate and forecast volatility, we can filter out the stock to reduce the risk by managing highly volatile stocks in the portfolio.

- It is possible to trade volatility by forecasting it and by using multiple strategies.

Volatility can be estimated using the following methods -

GARCH Model

- The generalised autoregressive conditional heteroskedasticity model is a method for estimating volatility.
- It is one of the most popular methods to forecast volatility.
- It weighs more recent observations more heavily, so the effect of the jump subsides.
- For example, the Exponentially Weighted Moving Average (EWMA) is generally between 0.9 and 0.99.

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda) r^2$$

- GARCH adds a long-term level in the above equation.
- GARCH(1,1) can be defined as -

$$\sigma_t^2 = \gamma V + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2$$

Looking at the context

While forecasting, it's useful to look at the ranges instead of just the current value.
For example - Measure volatility for periods of 20, 40, 80 and 100 days.

The following snapshot shows the volatility for multiple periods for Netflix from 2016 to 2021

	20 Day Vol	40 Day Vol	60 Day Vol	80 Day Vol	100 Day Vol
maximum	0.89	0.73	0.63	0.59	0.56
75th Percentile	0.47	0.47	0.45	0.46	0.45
Median	0.33	0.37	0.39	0.39	0.38
25th Percentile	0.27	0.29	0.32	0.33	0.33
Minimum	0.14	0.15	0.18	0.19	0.20

By comparing the forecasted volatility with the different quartiles of historical volatility, the forecasted value can be put in context to understand if the volatility is high or low. For example, if the forecasted value of the volatility is 0.5, this can be compared with different quartiles of historical volatility and can be concluded that the model forecasted higher volatility since the forecasted value is greater than the 75th percentile of forecasts for multiple periods.

Additional resources

- "Options, Futures and Other Derivatives" by John Hull.
- "Know Your Weapon" by Espen Haug
- "Options Trading" by Euan Sinclair