Option
Trading
Session Three: Options Pricing Models, Volatility Measurement and Forecasting

This is an adapted rendition of Dr. Euan Sinclair's lecture notes

Session Three Overview

- Options basics and terminology.
- Model independent features of options: arbitrage relationships between various options, and options and underlying.
- Option pricing variables and parameters.
- A toy binomial pricing model.
- Risk-neutrality.

Session Three Overview

- Properties of BSM solution.
- Deriving the Black-Scholes-Merton PDE.
- Options Greeks.
- Volatility measurement and forecasting.

Options

- An option gives the holder the right but not the obligation to do something.
- A call confers the right to buy the underlying at a certain price, the strike price.
- A put confers the right to sell the underlying at a certain price, the strike price.

Options

- Example: S=$100, a 90-strike call happens to be worth $14.
- $10 dollars of this is intrinsic value.
- The remaining $4 is extrinsic or time value.

- The intrinsic value is the amount by which the strike price of an option is profitable as compared to the stock's price in the market.
- Options also have “time value” (or “extrinsic value”) which reflects the fact the something might happen in the future to make the option intrinsically valuable.

Options

- In-the-money (ITM) means an option would have value if it expired right now. So, a call with a strike lower than the underlying price is ITM.
- We call this amount as 'Intrinsic Value'

- An out-of-the-money (OTM) option has no intrinsic value.
- An at-the-money (ATM) option has a strike equal to the underlying price (or sometimes the forward price so check in specific cases).

Options

- European options can only be exercised at a single time, the expiration date.
- American options can be exercised at a time prior to the expiration date.
- Almost all exchange listed options are American.
- The concept of European options was invented by Black and Scholes, because they had an analytic solution in this case, but these fixed-term options weren't like the ones that actually traded in America.

Options

- There are also Asians, which settle to an average.
- Russians, which are eternal lookback options.
- Bermudans, which can be exercised at a set of specific dates.
- Parisians, which are barriers that kick in when underlying has spent a certain time beyond the barrier.
- Hawaiians, an Asian option with an American exercise feature.

Exotic Options

- Many other exotic options “exist”.
- But most rarely trade or have never even been traded at all.
- Mainly an example of the worst kind of financial mathematics.
- Even when they exist, their “trading” generally consists of one-time selling to a customer (at an enormous markup).

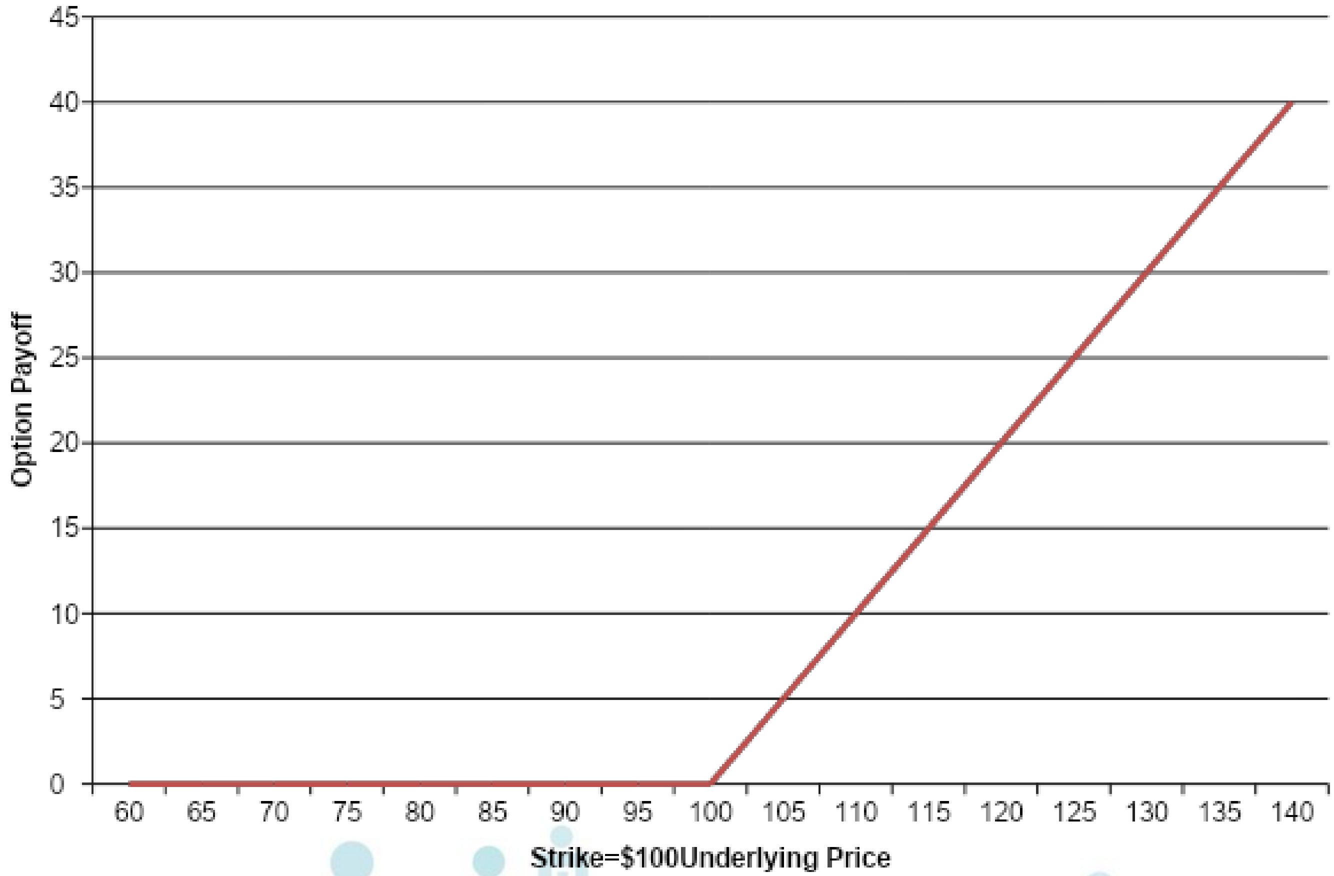
Options and Trading

- General rule: simplest products trade in the most complicated ways.
- Futures: market makers use many “tricks” such as book stacking, spoofing, flipping and stop hunting.
- Vanilla options: capturing edge is easy and traders are differentiated by risk management.
- Exotic Options: edge comes from structuring, legal and sales.

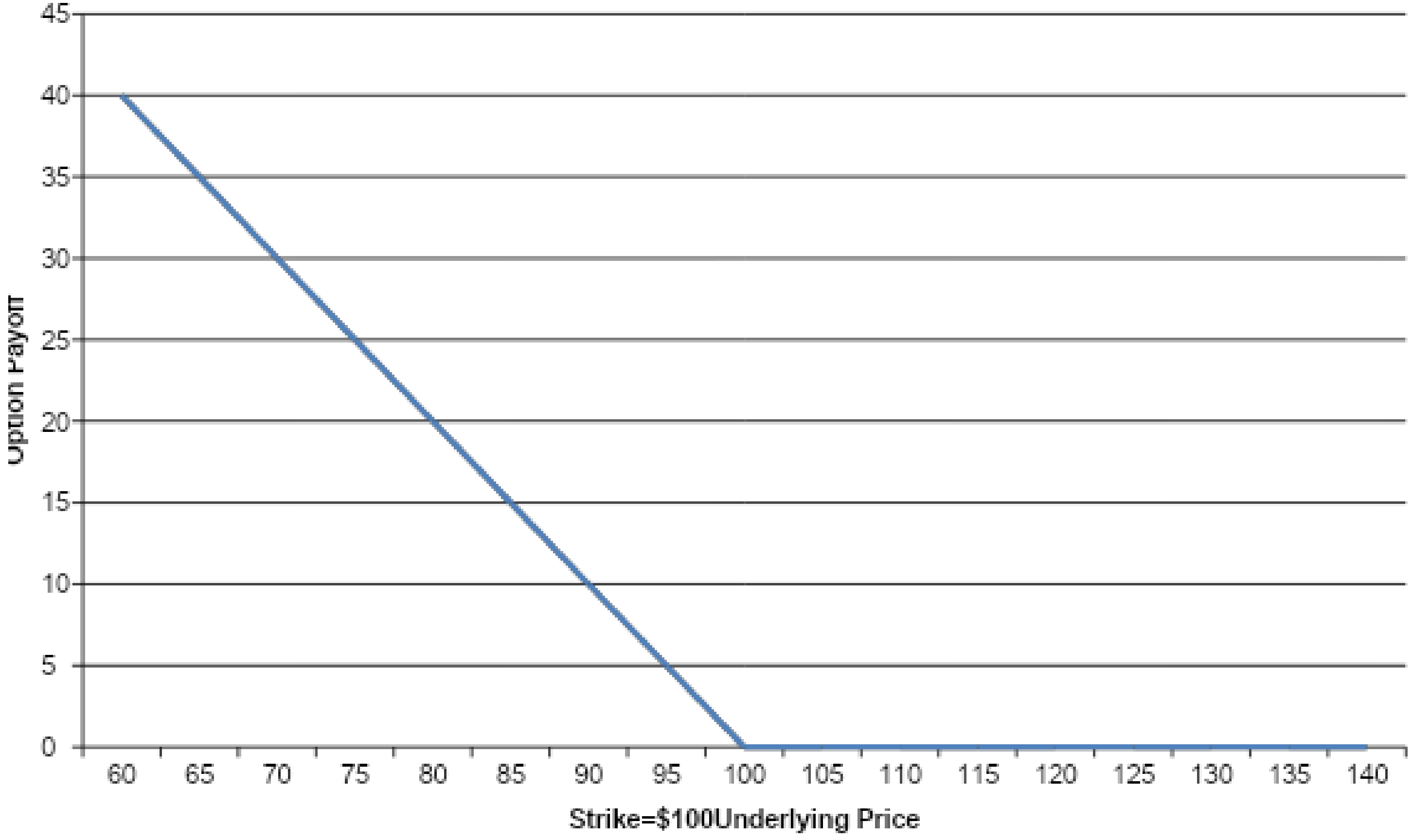
Option Payoff Functions

- The payoff is the value of the option at expiration.
- If the underlying price is S and the strike is X:
- Call payoff is MAX(S-X,0)
  - So, the holder makes money if the stock goes above the strike.
- Put payoff is MAX(X-S,0)
  - So, the holder makes money if the stock goes below the strike.

Payoff Diagram: A Call



Payoff Diagram: A Put

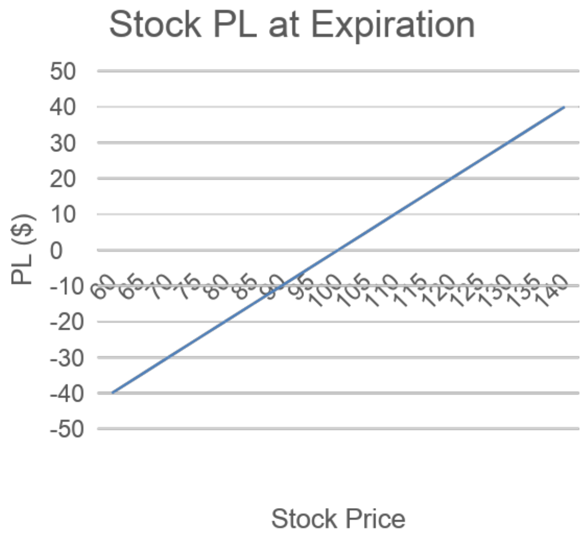


Combinations

- The piecewise linearity of options means it is easy to construct combinations of puts and calls to express any view.
- Aside: Be very careful about overusing this feature.
- Being right on a single view is hard enough.

Trade Example: Hedging

- Using options to limit downside losses.
- Example: long a share at $100. Pay $1 for the 90 put.



- Losses are guaranteed to be capped, unlike a sell order.

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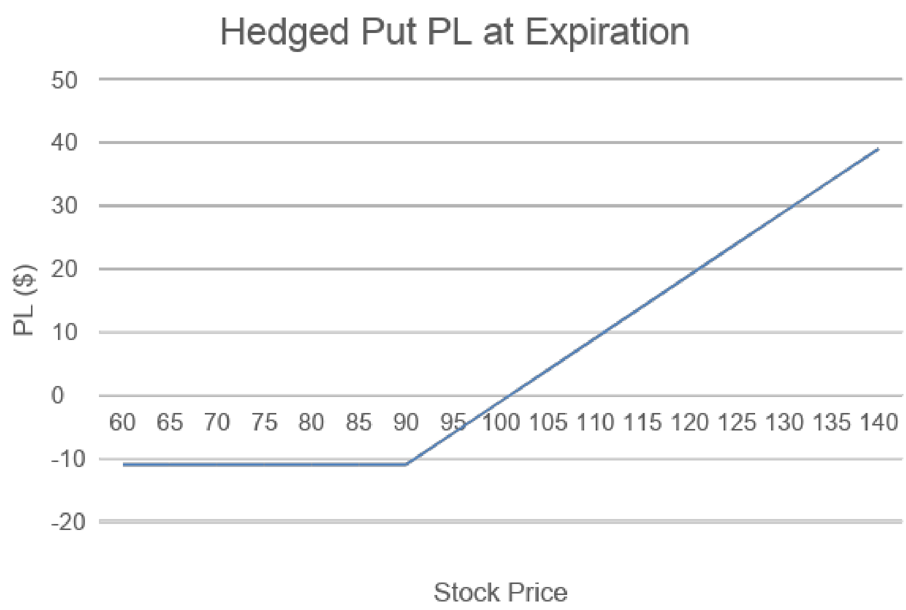

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Directional Option Trading

Pros:

- Limited downside (you can lose only the premium paid for option).
- Unlimited upside (literally, in the case of calls, and practically in the case of puts).

v 10.1.0

Directional Option Trading

Cons:

- Trade has limited time to work out.
- Can choose wrong strike.
- High transaction costs.
- Usually pay more than fair value.
- (Can really only manage the first two.)

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Strike Choice

- Return is highest when you pick the strike that only just expires in the money.
- But:
  - We don't know what that will be.
  - If we are wrong the option expires worthless.

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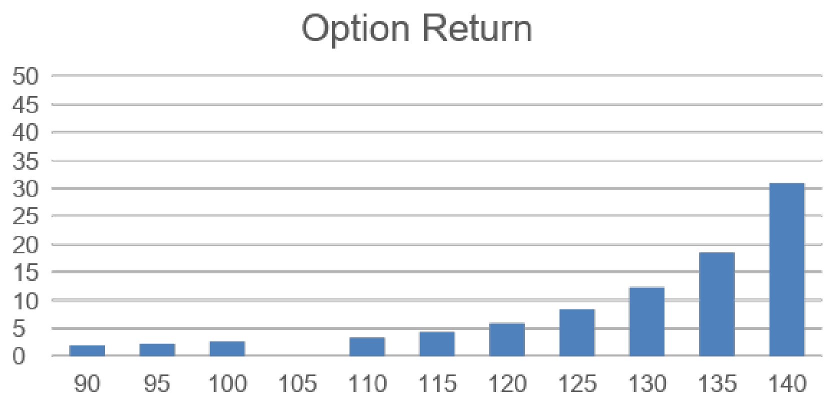
Strike Choice

- Example: one-year options on a $100 stock.
- Average return is 15% and volatility is 10%.



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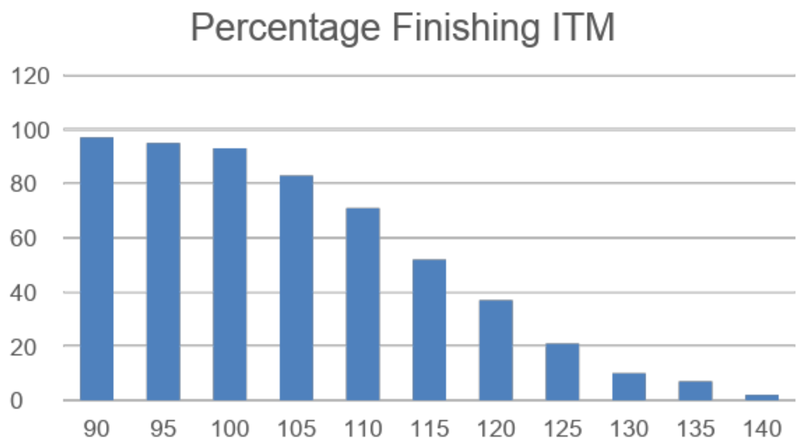
Strike Choice

- But higher strikes are more often worthless.



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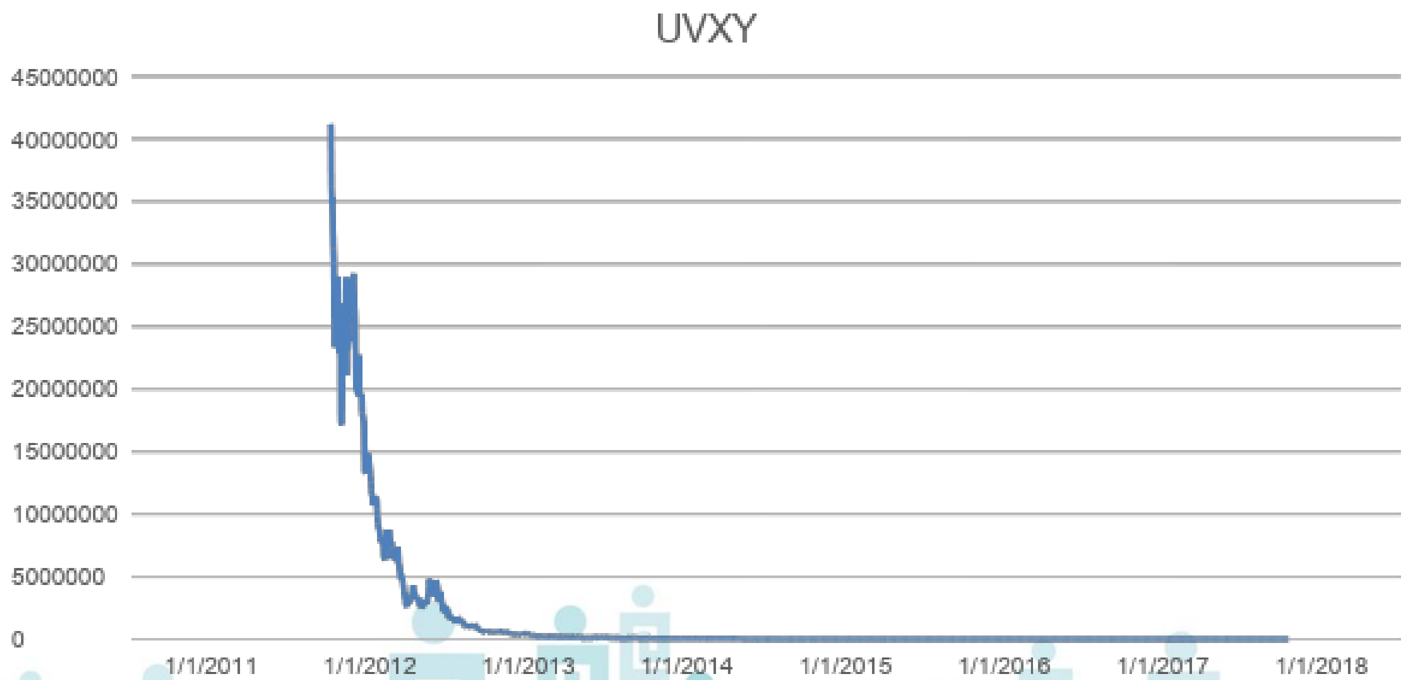
Trade Example: Speculation

- Simplest use of options is as a limited downside directional bet.
- UVXY is an ETN that returns 1.5 times the daily return of notional 30-day VIX futures.



Leveraged ETF Research

- Long Term Performance of Leveraged ETFs (https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1344133)
- Path-Dependence of Leveraged ETF Returns (https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1404708)

Trade Example: Buying a Put

- Clearly UVXY tends to go down, but why?
- (A reason is as important as an effect. If we don't know the reason, we won't know when the effect might stop).
- First reason is leverage.
- Imagine VIX: 100->110->99.
- UVXY: 100->115->97.75
- On average, leverage causes decay and this is exacerbated by volatility (the VIX futures are very volatile).

Trade Example: Buying a Put

- Second reason is the trading needed to keep constant maturity.
- Imagine 20-day future is 10 and 40-day future is 20.
- UVXY holds 50% of each to give a 30-day synthetic future exposure.
- But the next day (19-day and 39-day futures) we need to hold 45% in the front future and 55% in the back.
- So, each day, we sell something for 10 and buy something for 20…
- Depends on contango.

Trade Example: Buying a Put

- Note that even if UVXY drops, I can lose money.
- A “fair priced” option (defined later) has an average profit of zero, but a median loss.
- The option premium buys us limited losses and massive profit potential.
- Limited losses is why a long put might be a better trade here than a short position in the UVXY ETN.

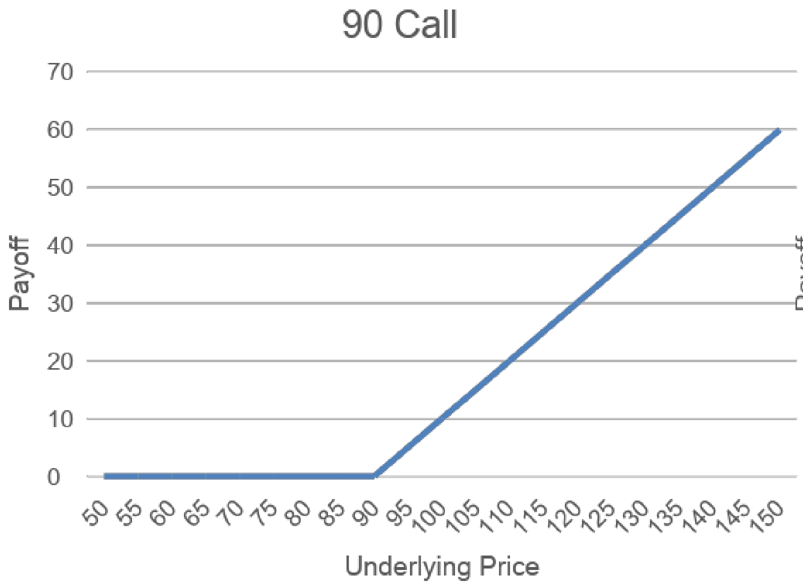
Trade Example: Buying a Put

- April 4th, 2021: UVXY is $5.24
- April 30th puts :

Strike	Offer Price	Break Even
4.0	0.12	$3.88
4.5	0.27	$4.23
5.0	0.53	$4.47

Example: Call Spread

- A long call spread is made by buying a call and selling a call with a higher strike and the same expiration.



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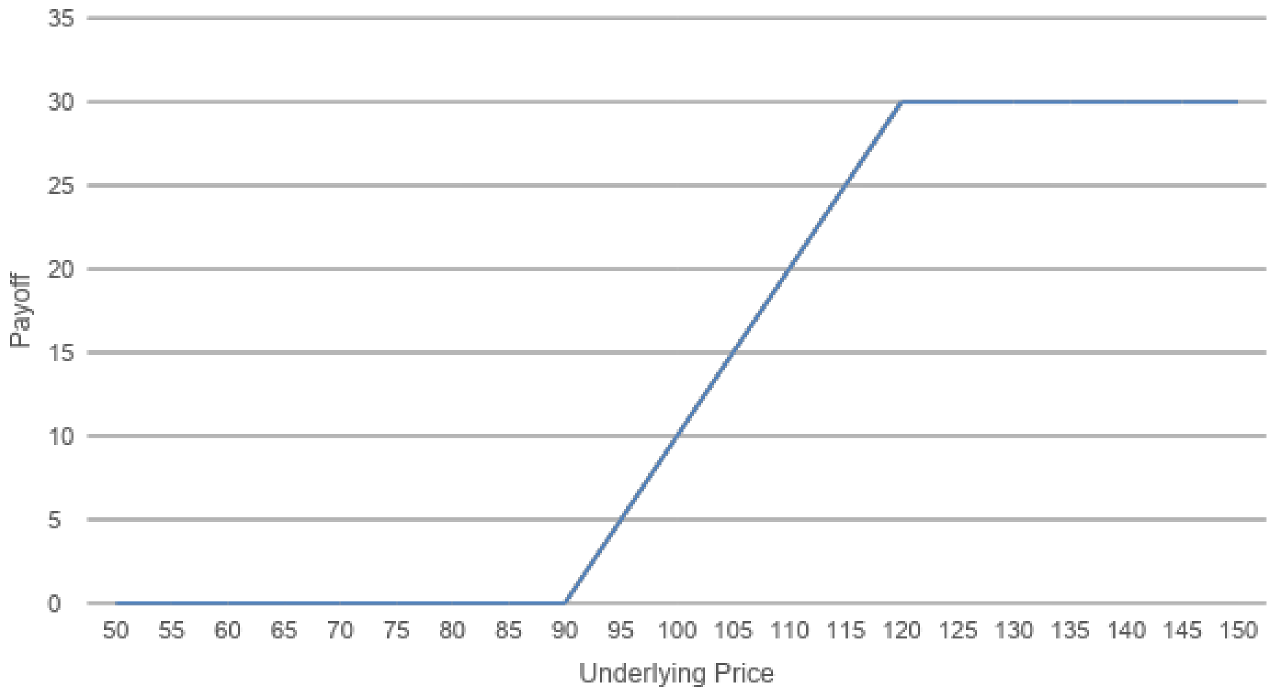
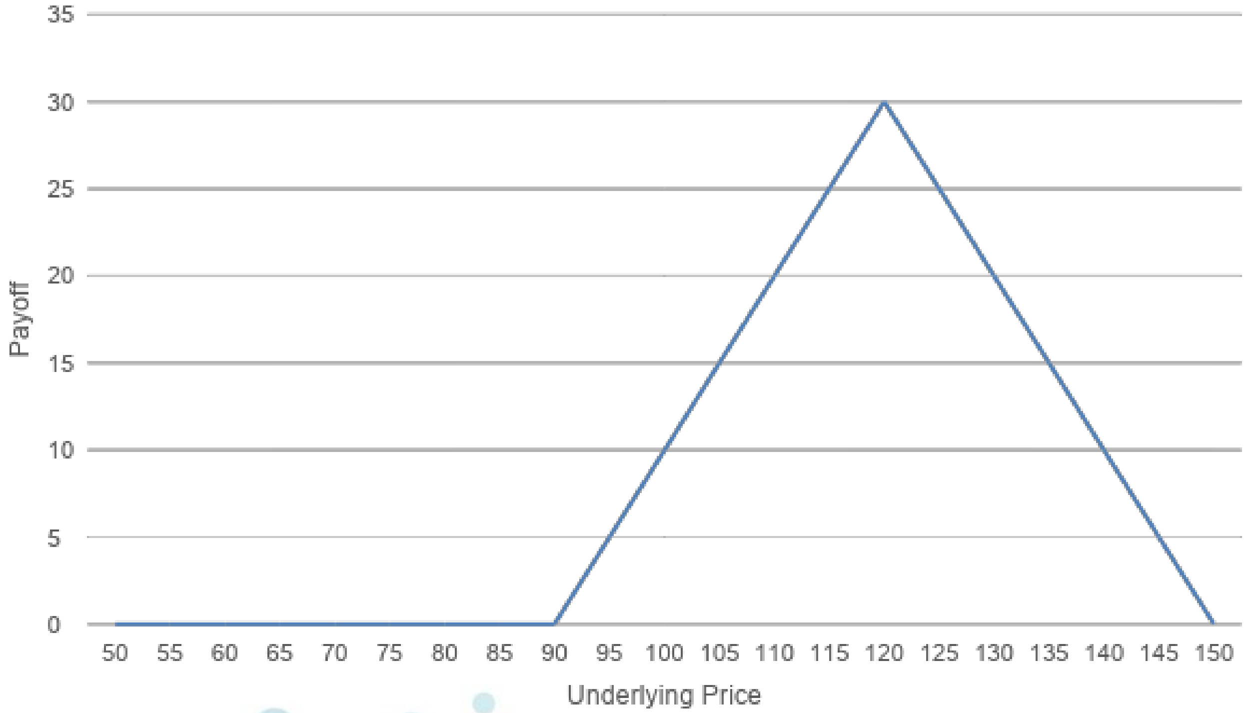


Example: 1 by 2 Call Spread

- A long 1 by 2 call spread is made by buying a call and selling two calls with a higher strike and the same expiration.

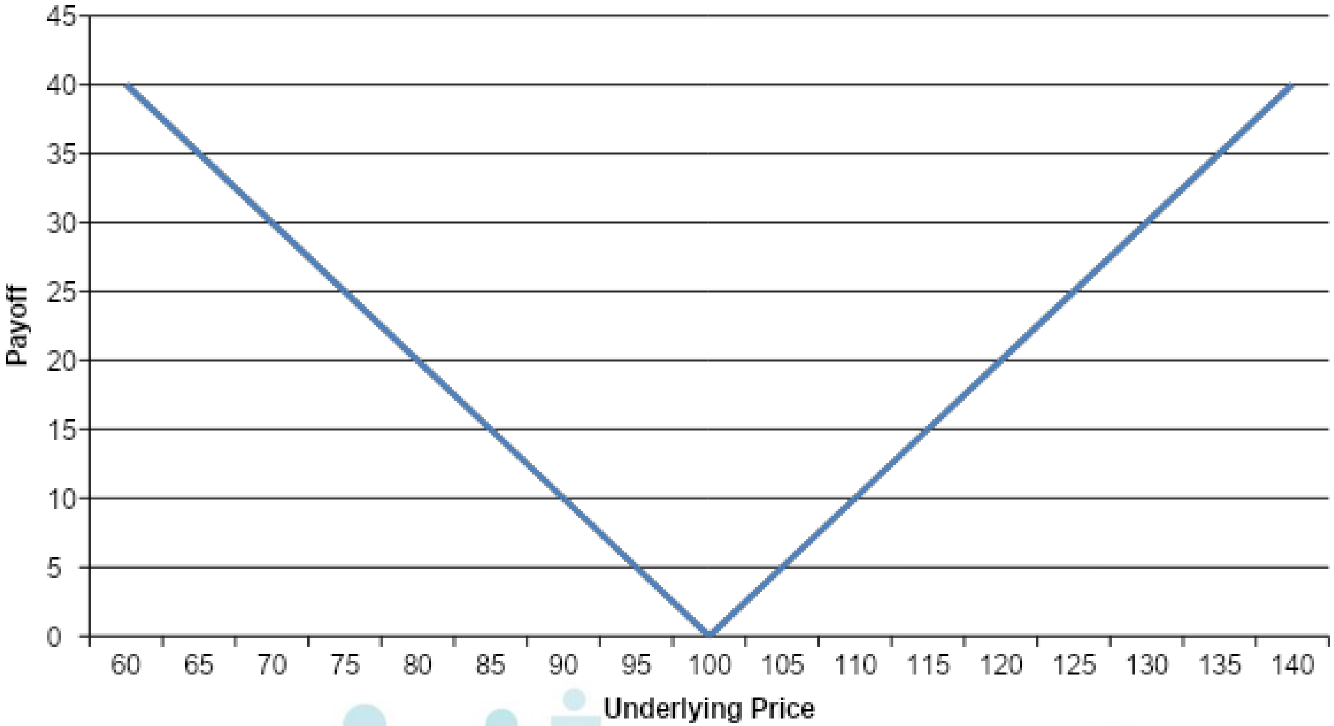


Example: Risk-Reversal

- A risk-reversal is made by buying a put and selling a call with a higher strike and the same expiration.

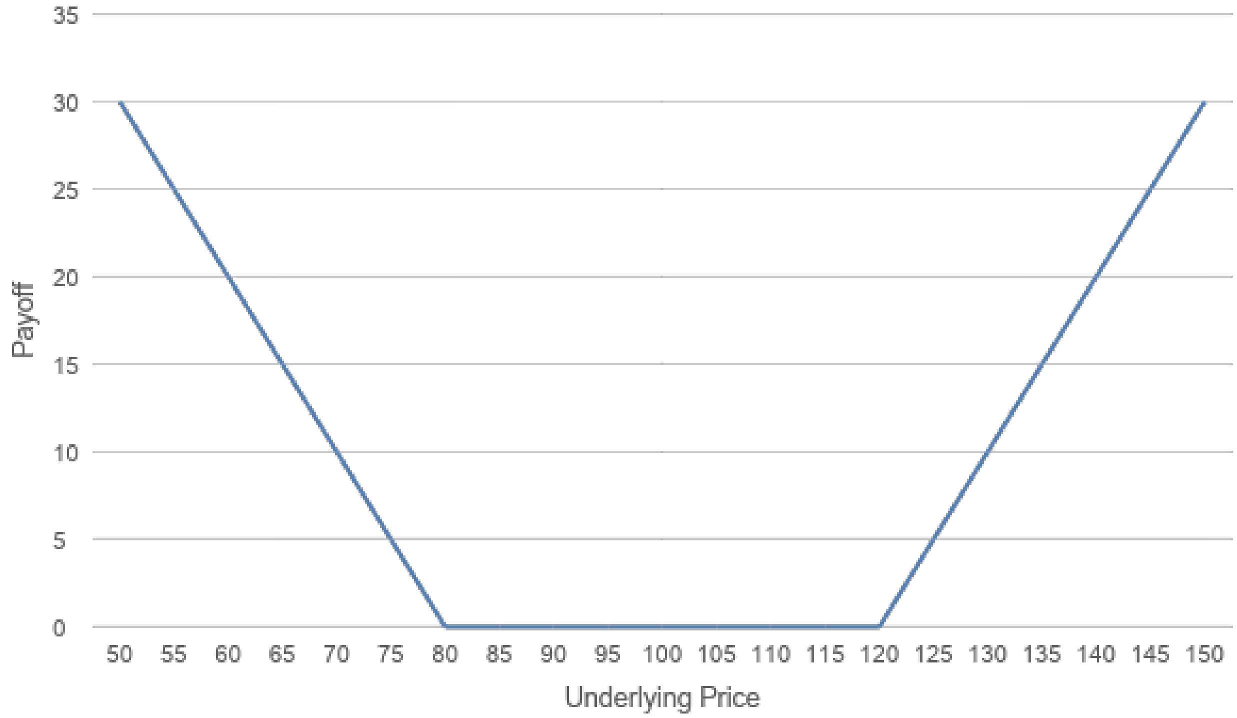
Example: Straddle

- A long straddle is made by buying a put and a call with the same strike and expiration.



Example: Strangle

- A long strangle is made by buying a put and a call with different strikes and same expiration.



Model Independent Relationships American>European

- American options are worth more than Europeans.
- If you don't choose to exercise an American option IS a European option.
- So, it is European plus some extra value.

Longer Dated American Options> Short Dated

- If you feel like it, you can turn the longer dated option into the short one by exercising.
- So, it IS the short-dated option plus some more optionality.
- Usually but not always true for European options (in high-rate environments getting cash sooner is better so a short-dated put can be worth more).

Call<Underlying

- If you exercise, you get the underlying.
- So, if C>S just sell the call and buy the underlying.
- At expiration our profit is C-expiration value of call + profit on stock.
- Above the strike this is $C - (S-X)+(S-S_0)=C+(X-S_0)>X$
- Below the strike this is $C-0+(S-S_0)>0$

Call<Underlying

- Example: 100 strike call on $100 stock is trading for $102.
- Sell call and buy stock.
- Assume we expire at $110. Call exercised.
- Lose $10 due to exercise, keep $102 call premium, gain $10 from our stock purchase.
- Make $102.
- Assume we expire at $90.
- We lose $10 on our stock purchase but keep $102 from call sale.

Put<Strike

- If you exercise, you get the strike minus the current price.
- So, the highest value the option can ever have is the strike (if S=0).

- Example: 100 strike put on $100 stock is trading for $102. Sell the put.
- Expire at $0. Lose $100 on put exercise but keep $102 put premium.
- Expire above the strike, keep $102 premium.

Lower Strike C>Higher Strike C

- There will be times when a lower strike call can be exercised, and the higher strike call cannot.

- Example: 100 strike call is trading for $2 and $110 call is trading for $5.
- Sell 110 call, buy 100 call and collect $3.
- Expire at $90, both calls are worthless, and we keep $3.
- Expire at $105. Make $5 on the 100 call and keep $3 premium.
- Expire at $120. Make $20 on 100 call, lose $10 on 110 call and keep $3 premium.

- Opposite is true for puts.

Put/Call Parity (very important)

- Puts and calls are “the same thing”
- Specifically

$$C-P=S-X\ e^{-rT}$$

- Proof: Form a portfolio of long put, short call and long one share.

Instrument	Initial Value	Value at Expiration	
		If S<X	If S> X
Long P(X)	P(X)	(X-S)	0
Short C(X)	-C(X)	0	-(S- X)
Long Stock	S	S	S
Total Portfolio	S+ P(X) -C(X)	X	X

Put/Call Parity: Example

- This concept was used to avoid usury laws in Europe in the middle ages.
- I want to borrow $100 and the counterparty, Anne, wants to charge 100% interest, an illegally high rate.
- I sell her a bike for $100 (the loan) and a 200-strike put.
- She sells me a 200-strike call.
- If the bike doesn't become worth more than $200, she exercises her put and I get the bike back (for a loss of $100).
- If the bike's value increases above $200, I exercise my call instead.

Put Call Parity Implications

- A call is like borrowing money to buy stock and a put. I.E it is a limited downside leveraged position in the underlying.
- The payoff from any instrument can be replicated with the other three.

Put Call Parity Implications

- For call and put to have equal value the strike price must be the forward price.

$$C-P=S-X\,e^{-rT}$$

- Assume C=P

$$0=S-X\,e^{-rT}$$

$$X=S\,e^{rT}$$

Put Call Parity Implications

- The relationship does not hold for American options, because we can't be sure of the duration of a short option position.

Put/Call Parity and Americans

- For European options

$$C\text{-}P=S\text{-}X\,e^{-rT}$$

- Example: S=0, X=100, T=1, r=0.1
- C=0, P= 90.48 so C-P=-90.48 (from the BSM model)
- $S\text{-}X\,e^{-rT}$=-90.48
- But for Americans, C=0 and P=100, or else we just exercise, hence P/C parity equation fails.

Put Call Parity Implications

- If we can make a bullish option position into a bearish option position by adding stock, then we should also be able to directionally neutralize an option.
- So, P/C parity is an important precursor to volatility trading.

Option Pricing: Inputs

- Underlying price.
- Strike: distance between strike and price define the option.
- Rates : rates are the discount factor for cash flows.
- Dividends: income from the stock which affects its future value.
- Time.
- Volatility.

Option Pricing: Inputs

- Why not return?
- Surely if μ>0 calls will be worth more and puts less?
- Refer back to put/call parity.

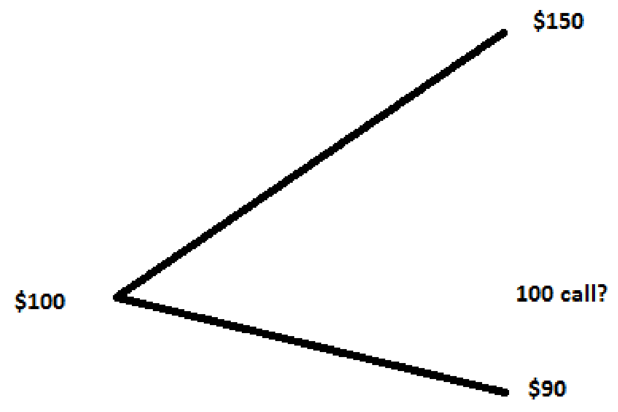
$$C-P=S-X e^{-rT}$$

- If you believe a positive drift leads C to increase, it also means P must increase.
- So positive drift leads both puts and calls to increase?
- Contradiction.

A Toy Model: One Period Binomial

- We have a $100 stock that in the next period can go up to $150, or down to $90.

- What is the 100-call worth?



A Toy Model: One Period Binomial

- We sell the call and hedge the directional risk by buying h shares (we don't know what h is yet).
- So, portfolio value is

$$S \times h - C$$

A Toy Model: One Period Binomial

- At \$150, our portfolio is worth \$150 h -\$50.
- At \$90, our portfolio is worth \$90 h.
- For us to be hedged, we need to be indifferent to these states so
- \$90 h =\$150 h -\$50 or h=5/6.
- At \$150, our portfolio is worth \$150 x (5/6) - \$50=\$75
- At \$90, our portfolio is worth \$90 x (5/6)=\$75

- So initial fair value, Sxh-C=final value
- (5/6) x \$100 –C=\$75 or C=\$8.33

A Toy Model: One Period Binomial

- Note this is *not* the probability weighted average of the value in each state.
- We don't even know what the probabilities are.
- So, the expected return of the stock doesn't matter (in accordance with our intuition).
- But the spread of the stock results does matter (check this).

BSM from P/L Analysis

- Again, we form the portfolio:
  - Start with no cash
  - Buy a call option.
  - Sell some stock, *h*, as a hedge.
  - Put any remaining cash in the bank (or borrow it).
- Because we've just swapped stuff around our net worth is zero.

BSM from P/L Analysis

- In a small time-step, *dt* our portfolio value changes due to three things:
  - The call changes value.
  - The stock moves.
  - We get (or pay) some interest.

BSM from P/L Analysis

- The call changes by its theta (time decay):

$$\theta\delta t$$

- Our interest income is on the capital, *hS*, that we got from selling stock and –*C*, that we got from buying the call. This is:

$$r(hS - C)\delta t$$

BSM from P/L Analysis

- The effect of the stock move is more complicated:

$$P = C - hS$$

- So, the change in *P* is:

$$C_{S_0+dS} - h(S_0 + dS)\text{-}C_{S_0} + hS_0$$
$$= C_{S_0+dS} - hdS - C_{S_0}$$
$$= C_{S_0} + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial C^2}{\partial S^2}dS^2 - hdS - C_{S_0}$$

- But to first order, $\frac{\partial C}{\partial S} = h$ so,

$$\partial P \approx \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\langle \Delta S \rangle^2 \delta t$$

BSM from P/L Analysis

- Variance is defined as the standard deviation of the (log) returns

$$\sigma^2 = \frac{1}{N}\sum(r_t - \tilde{r})^2$$

$$r_t \equiv ln\left(\frac{S}{S_{t-1}}\right) \approx \left(\frac{S - S_{t-1}}{S_{t-1}}\right)$$

$$\sigma^2 \approx \frac{1}{N}\sum\left(\frac{S - S_{t-1}}{S_{t-1}}\right)^2$$

- So, on average

$$(\sigma S)^2 \approx (S - S_{t-1})^2$$

$$\partial P \approx \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\langle\Delta S\rangle^2 \delta t$$

BSM from P/L Analysis

- Summing these, ignoring the common δt multiplier and realizing that this P/L must be zero, we get:

$$\theta + \frac{1}{2}\Gamma\sigma^2 S^2 + r(Sh - C) = 0$$

Or, relabeling *h* as Δ

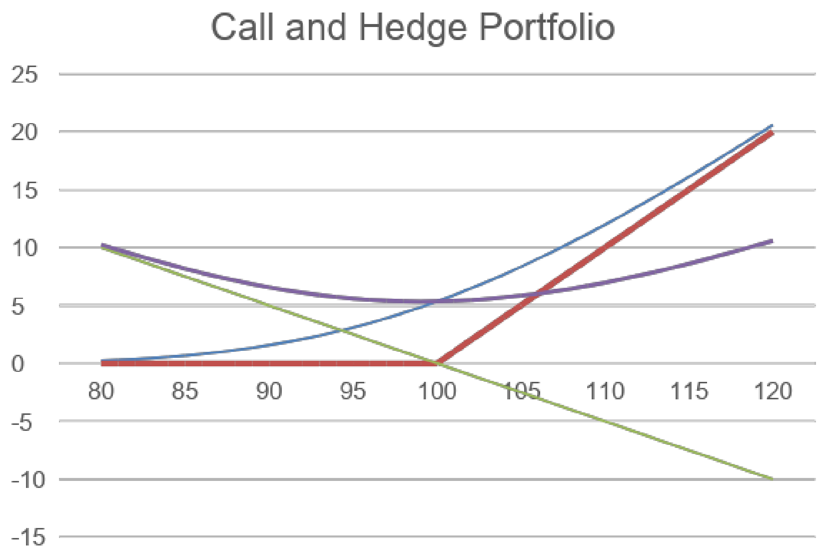
$$\theta + \frac{1}{2}\Gamma\sigma^2 S^2 + r(S\Delta - C) = 0$$

Wait a Minute…

- Why are we only considering second order terms in that Taylor series?

1. We could assume the returns only have non-zero terms up to variance.
2. If we are prepared to accept this only holds for very small times, we can ignore higher order terms (as long as they are finite).
3. The “appeal to a picture” argument.

Wait a Minute…



Wait a Minute…

- Why only first order time derivatives?

1. Time is of fundamentally different type to price (deterministic versus stochastic).
2. Effect of time is much smaller than price moves.

Example: For a typical, one-month ATM stock option, a 1% price move changes the price by 16%. This is the same as the time change over 9 days.

Solution for European Options

- The option price is essentially an interpolation between the stock price and the bond price. Below is 100 strike call.

Option Value

- I.T.O the equations

$$C = SN(d_1) - Xexp(-rT)N(d_2)$$

N() is cumulative normal distribution.

$$d_1 = \frac{ln\left(\frac{S}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)\text{T}}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Option Value

$$C = SN(d_1) - Xe^{-rT}N(d_2)$$

- This is an interpolation between stock, *S*, and the cash
- value, *X*.
- **If** $S \gg X, ln\left(\frac{S}{X}\right) > 1 \text{ so } ln\left(\frac{S}{X}\right) > \left(r + \frac{1}{2}\sigma^2\right)\text{T}\ and\ N(d_1) \sim N(d_2) \approx 1,$ **so** $C \approx S - X$
- **If** $S \ll X, ln\left(\frac{S}{X}\right) < 0$ **so** $d_1 \approx d_2 < 0 \text{ and } N(d_1) \sim N(d_2) \approx 0\ so\ C \approx 0$

Option Value

- The interpolation interpretation also tell us what the terms "mean".
- A call is a part ($N(d_1)$) of the stock and a part ($-exp(-rT)N(d_2)$) of the cash/bond.
- $N(d_1)$ is the amount of stock we are exposed to.
- $N(d_2)$ is the amount of cash we are exposed to. i.e. what we need to deliver at expiration (remember the strike, X, is the cash we need to pay)

Option Value

- $N(d_2)$ is the amount of cash we are exposed to. i.e. what we need to deliver.
- Alternatively, $N(d_2)$ is the probability we need to pay out on the option.
- $N(d_2)$ is the probability of finishing in the money.
- No!
- Remember all of this takes place in the *risk neutral* world.

Option Value

- This interpretation is only true in a risk-neutral world; a world that doesn’t exist.
- In the real world, the probabilities are mainly driven by drift, something we have no interest in (as a pricing variable) or significant ability to predict.
- Never use options to make probabilistic statements about the real prices.
- This has been tested. Options do not finish ITM at the rate predicted by $N(d_2)$.

Option Value

- A put can either be priced by using P/C parity, or else by solving the PDE with a different boundary condition.

$$P = Xexp(-rT)N(-d_2) - SN(-d_1)$$

“Greeks”

- “Greeks” is the collective term for partial derivatives (in the mathematical sense) of the option price.
- Can be calculated either analytically or numerically.
- Volatility traders generally quantify their positions i.t.o. Greek exposure.
- Because the BSM equation is linear, greeks can be aggregated across a position by adding the greeks of each individual option.

Delta

- Delta is the partial derivative of the option price w.r.t. underlying.

$$\Delta_{call} = N(d_1)$$

$$\Delta_{put} = -N(-d_1)$$

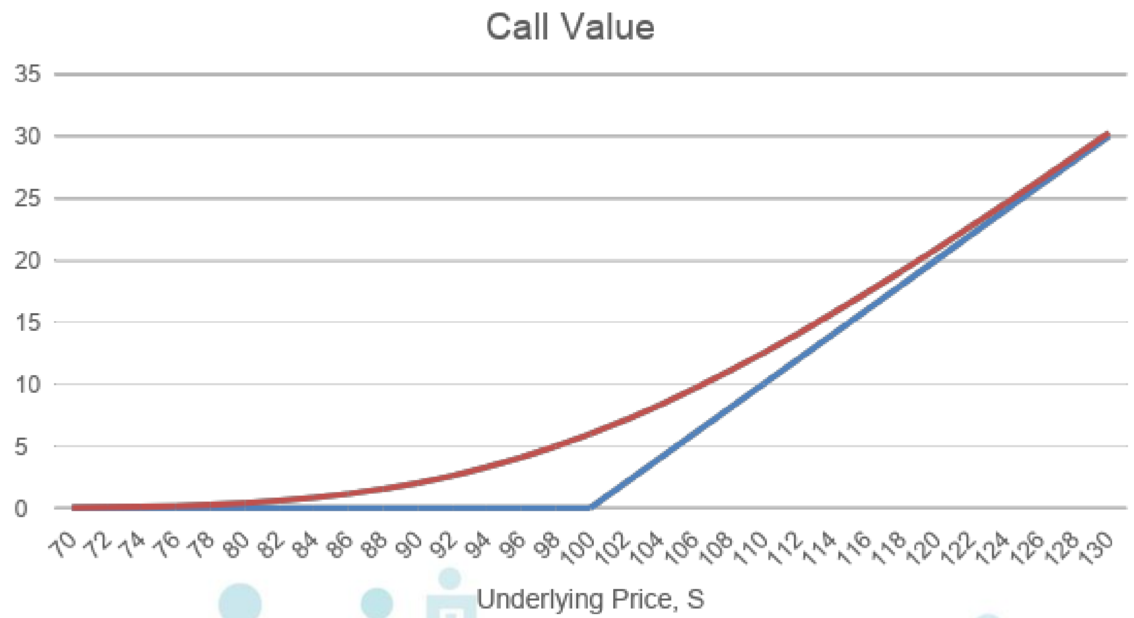
- P/C parity implies

$$\Delta_{call} - \Delta_{put} = 1$$

- So, a 90-delta call (0.9) will have the same strike as a 10-delta put (really -0.1)

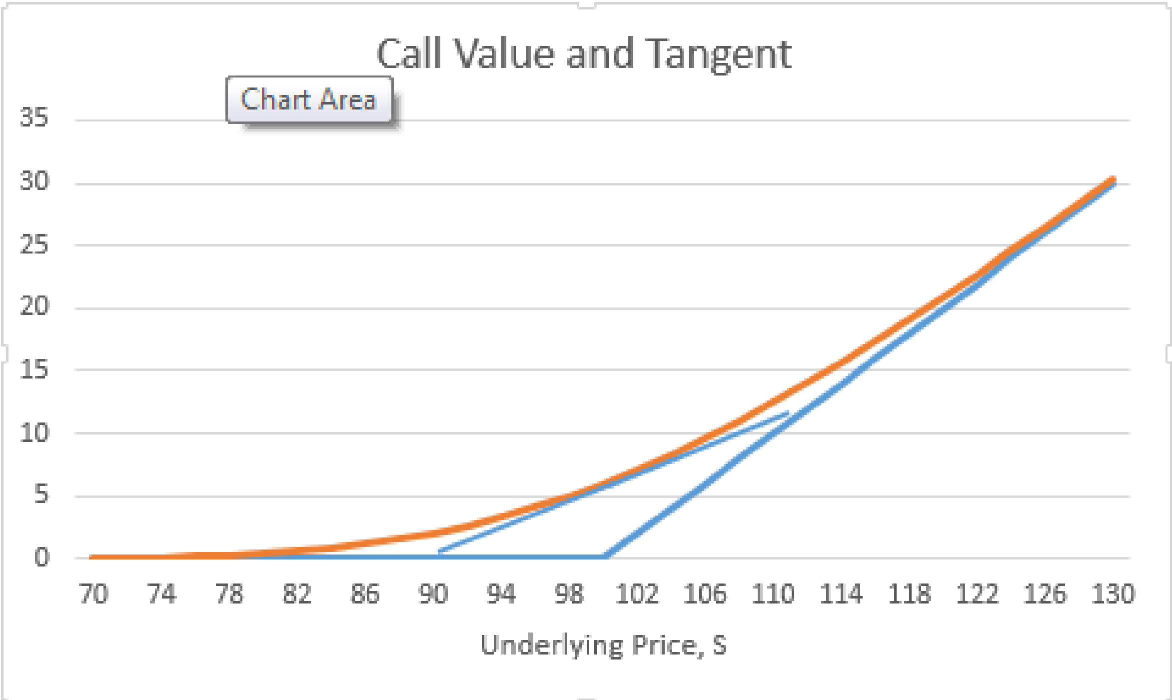
Delta (Graphically)

- Delta is the tangent of the option price Vs underlying graph.
- We have a call with a strike of 100.
- Expiration value (blue) and current premium (red) are shown below:



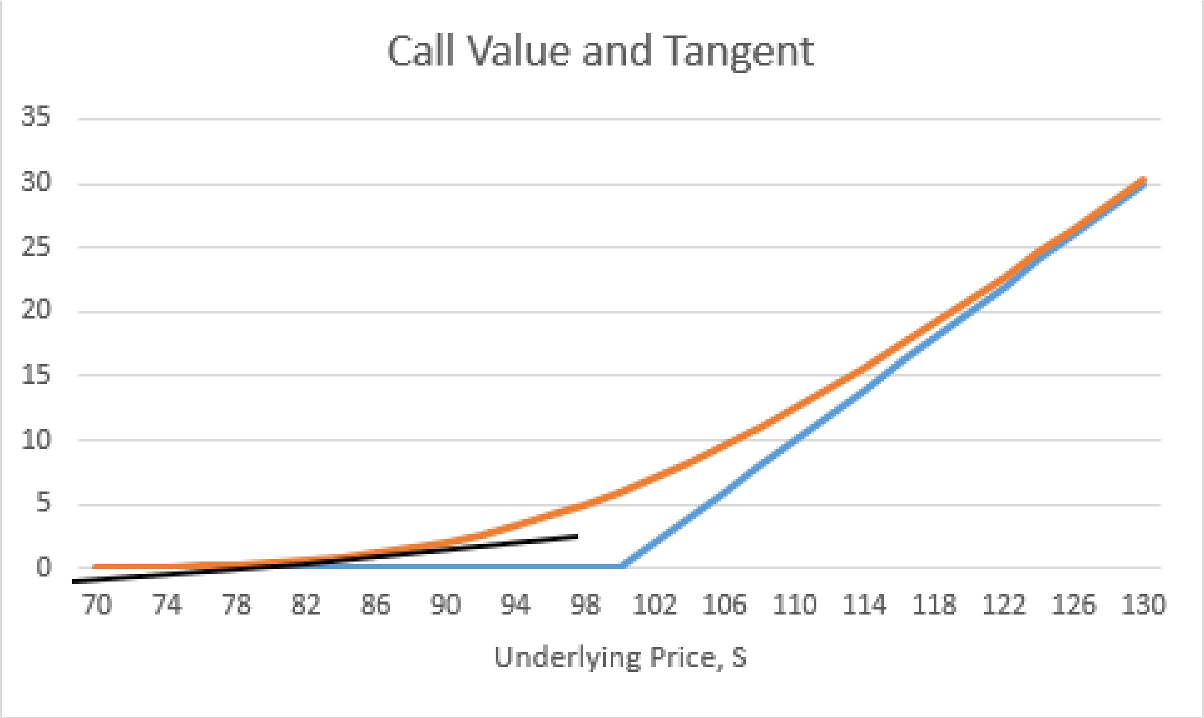
Delta (Graphically)

- Approximate the derivative by a difference.
- $\frac{\partial C}{\partial S} \approx \frac{\Delta C}{\Delta S}$
- This is tangent of the Call value.

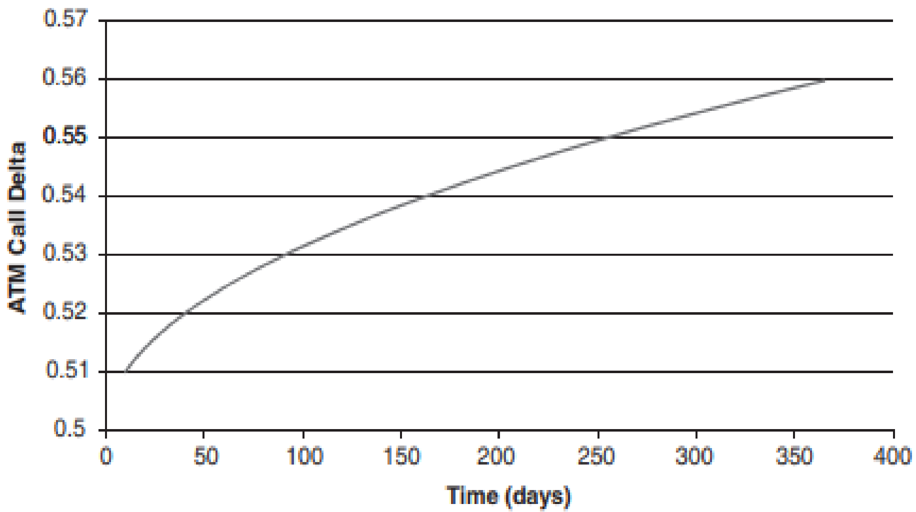


Delta (Graphically)

- If underlying price is much lower than the strike, the option is way out of the money and delta will be almost zero.
- Here is the delta line when S=80.



Delta vs Time/Volatility



- At infinite time (or volatility) call delta goes to one and put delta to zero.
- At high volatility, the stock can go much higher but is bounded below.

Delta Misconceptions

- Delta IS the partial derivative of the option price w.r.t. the underlying.
- Delta IS the hedge ratio.
- Delta IS NOT probability of finishing in the money. First, in risk neutral world this is $N(d_2)$. Second, risk neutral world isn't the real world.
- ATM delta is close to 0.5 but calls will be slightly higher.

Delta Misconceptions

- ATM call delta >0.5 and increases as volatility increases.
- Why?
- Normal returns mean percentage moves are of equal probability up or down.
- S=100, two 1% up moves 100->101>102.1 i.e., up $2.1
- S=100, two 1% down moves 100->99>98.1 i.e., down $1.9
- So as delta is the hedge-ratio we need to sell more shares to protect against a down move because it will be a smaller move.
- If volatility is much higher the effect is magnified, so delta of a call increases with volatility.

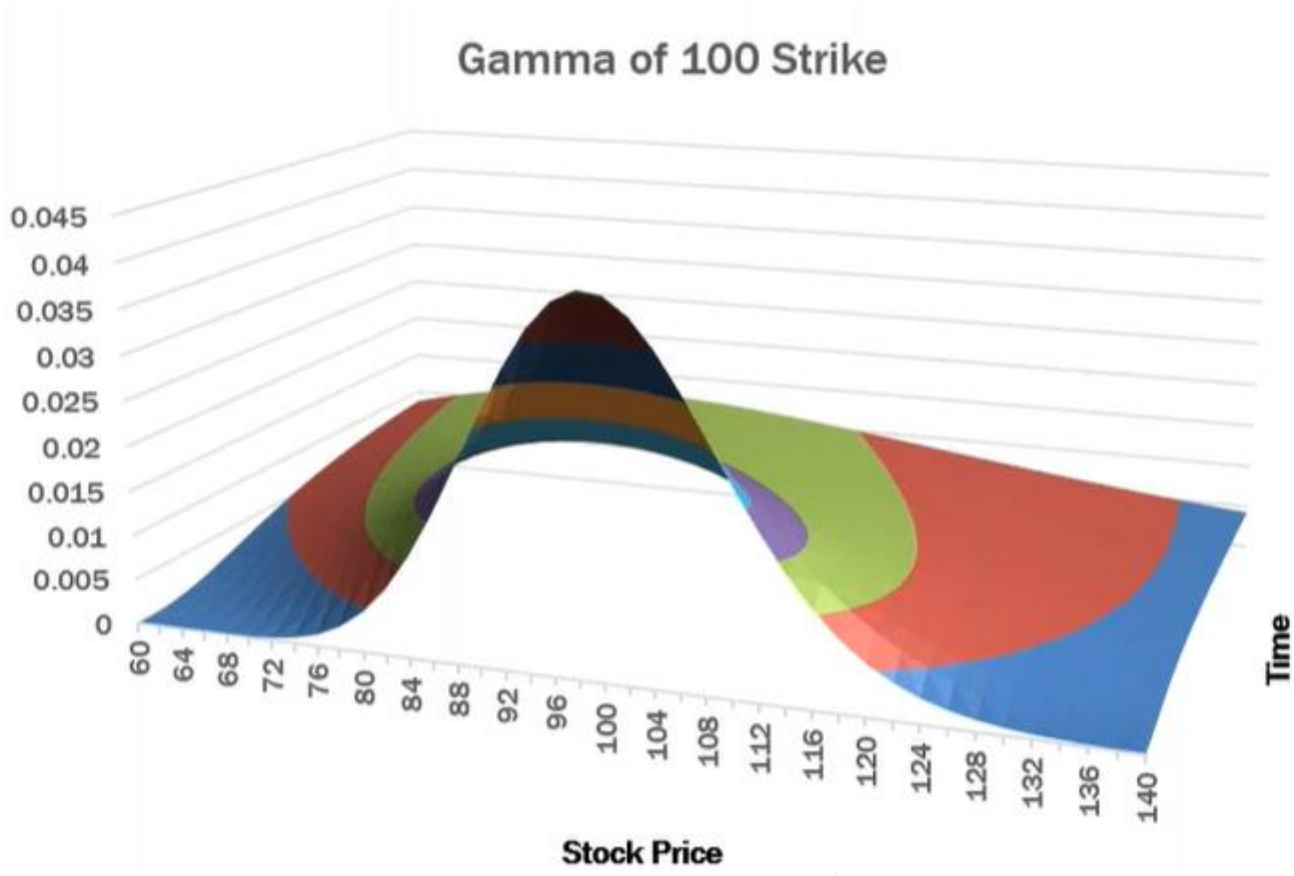
Gamma

- Gamma is the partial derivative of the delta w.r.t. the underlying.
- To manage delta, must know gamma.

$$\Gamma_{call} = \frac{n(d_1)}{S\sigma\sqrt{t}} = \Gamma_{put}$$

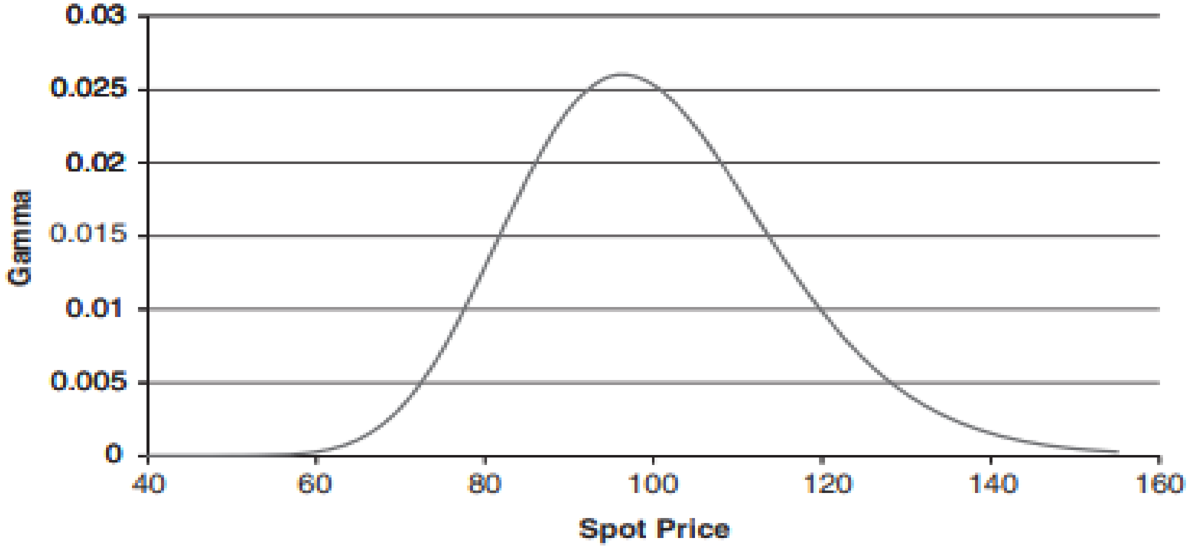
- Gamma is highest for ATM short-dated options (actually slightly below the actual ATM strike).
- Indeed, traders usually refer to short dated options just as “gamma”.

Gamma



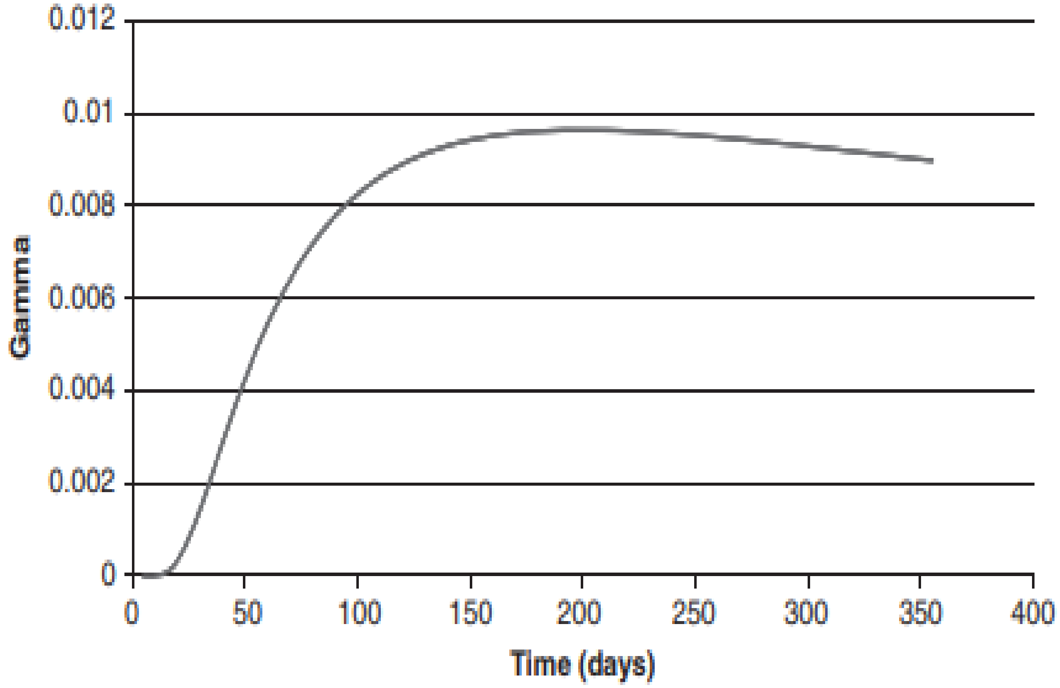
Gamma of 100 Strike
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
60
64
68
72
76
80
84
88
92
96
100
104
108
112
116
120
124
128
132
136
140
Stock Price
Time

Gamma V Stock Price

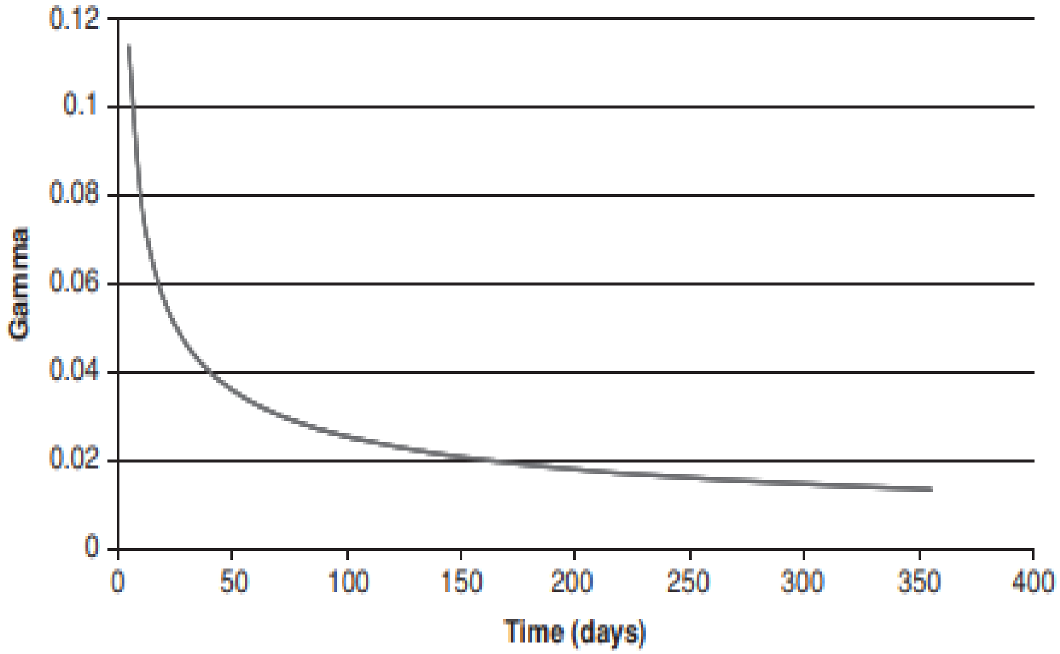


- Unsurprisingly, delta looks a lot like a cumulative normal distribution and gamma looks a lot like a normal distribution.

Gamma V Time/Vol Out of the Money Option



Gamma V Time/Vol At the Money Option



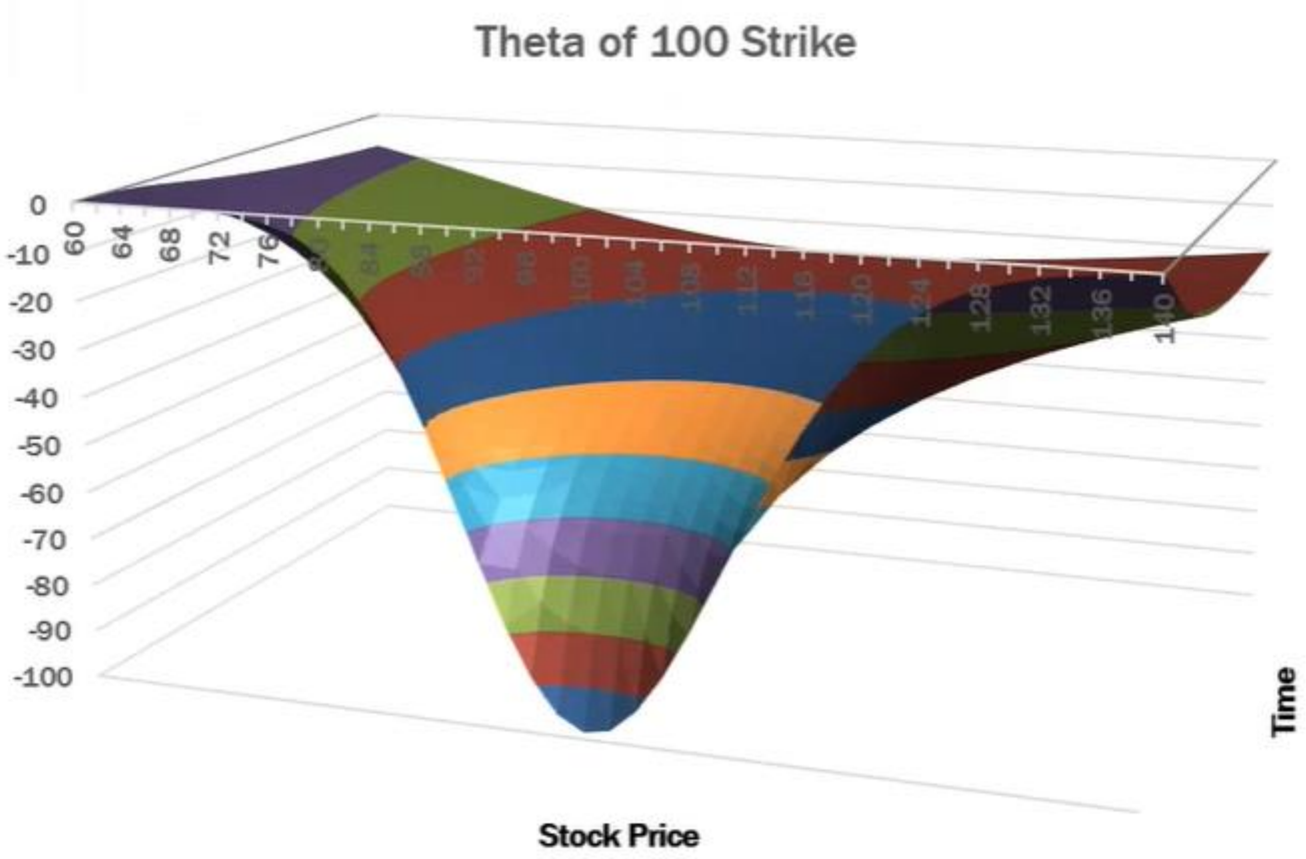
Theta

- Theta is the partial derivative of the option price w.r.t. time.
- Normally expressed in daily units.

$$\theta_{call} = -\frac{S\sigma n(d_1)}{2\sqrt{t}} - rXexp(-rT)N(d_2)$$

$$\theta_{put} = -\frac{S\sigma n(d_1)}{2\sqrt{t}} + rXexp(-rT)N(-d_2)$$

Theta



Theta of 100 Strike
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
60
64
68
72
76
84
92
96
100
104
108
112
116
120
124
128
132
136
140
Time
Stock Price

Theta

- First term (same for puts and calls) is the time decay, the amount the option declines because volatility has less time to act.
- Second term is often misunderstood (or just ignored).

$$\theta_{call} - \theta_{put} = rXexp(-rT)$$

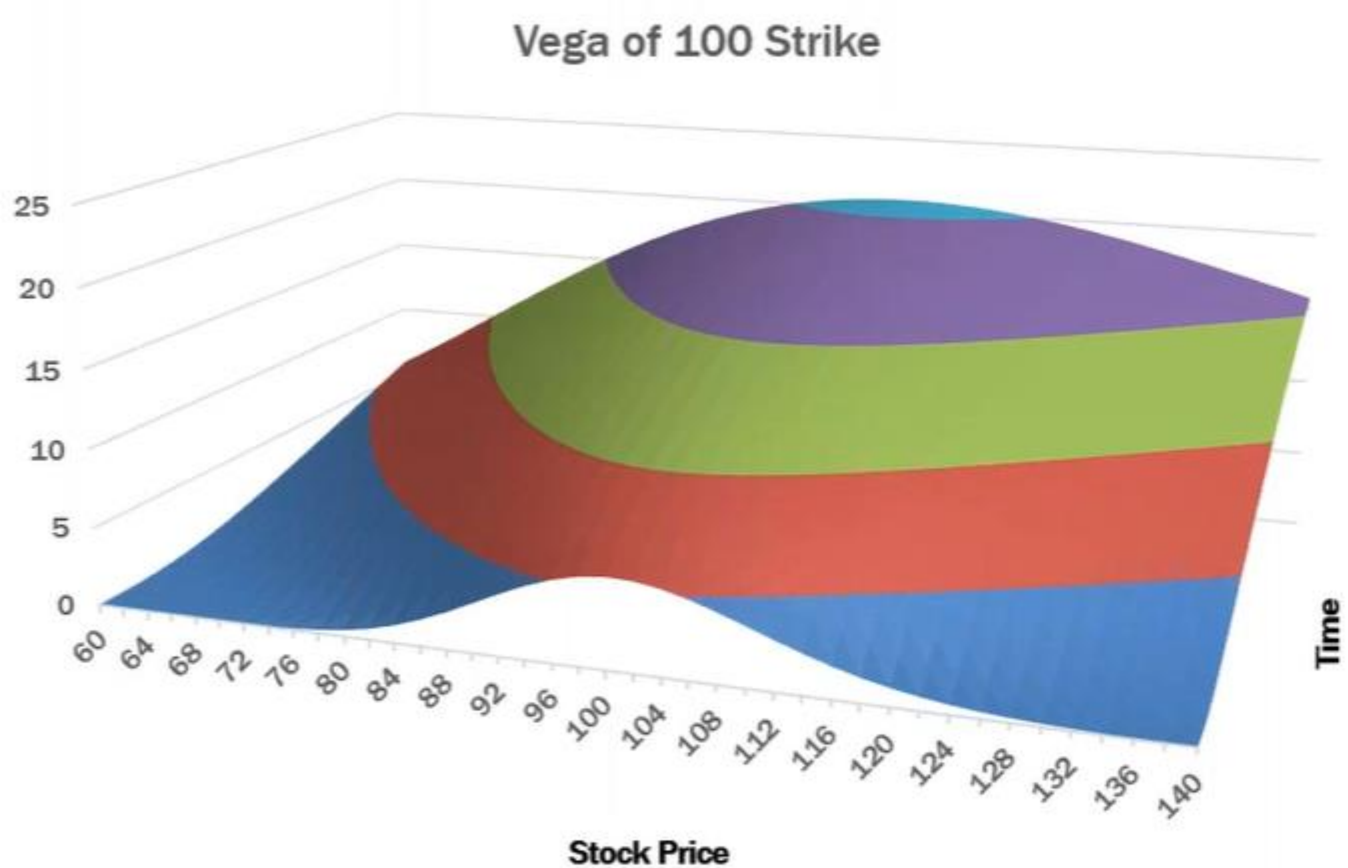
- From P/C parity discussion we know a long call and short put has no volatility exposure.
- But it does have carry, the cost of holding the strike value in cash.
- Theta just allocates this carry across calls and puts according to the chance of each finishing in the money.

Vega

- Vega is the partial derivative of the option price w.r.t. volatility.
- Inconsistent with the assumption of constant volatility, but very important to traders.

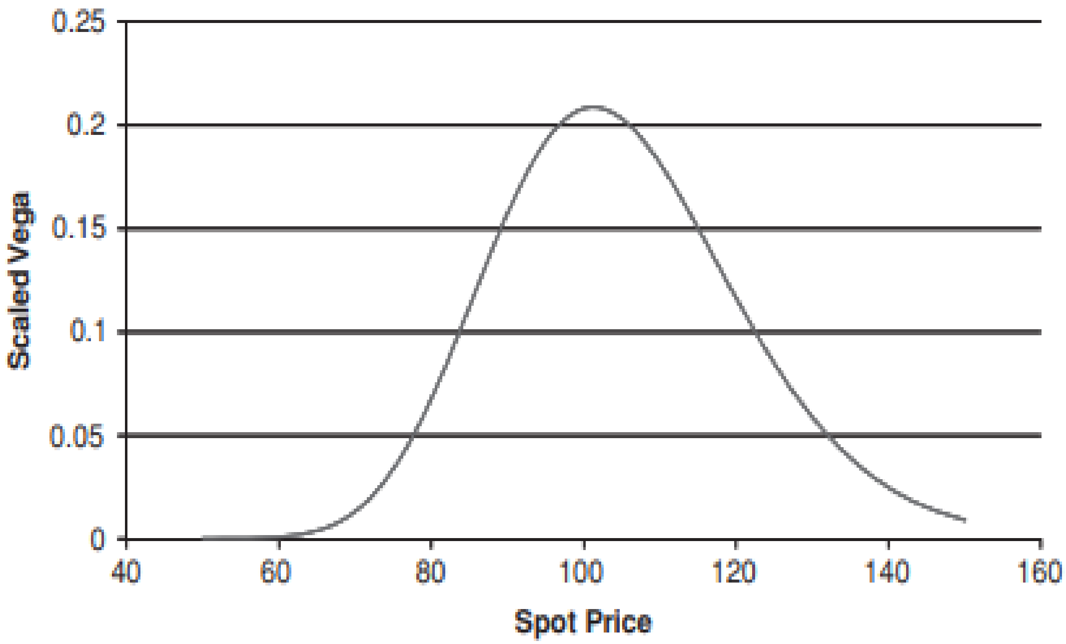
$$Vega_{call} = Sn(d_1)\sqrt{t} = Vega_{put}$$

Vega

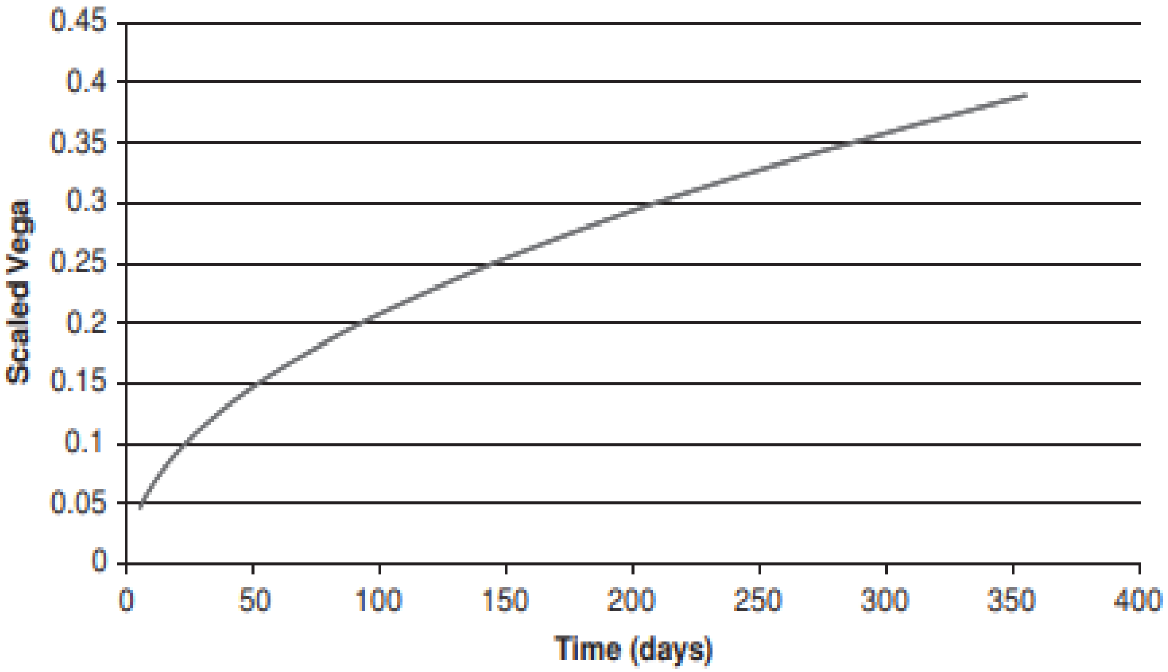


Vega of 100 Strike
25
20
15
10
5
0
60
64
68
72
76
80
84
88
92
96
100
104
108
112
116
120
124
128
132
136
140
Stock Price
Time

Vega Vs Spot


0.25
0.2
0.15
0.1
0.05
0
Scaled Vega
40
60
80
100
120
140
160
Spot Price

Vega Vs Time



0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Scaled Vega
0
50
100
150
200
250
300
350
400
Time (days)

Minor Greeks: Rho

- Rho is the partial derivative of the option price w.r.t. interest rates.
- Inconsistent with the assumption of constant rates.
- Usually ignored by traders, as it is managed at the firm level by treasury.
- Generally scaled to be dollar change for a 1% move in rates.

$$\rho_c = TXexp(-rT)N(d_2)$$

$$\rho_p = -TXexp(-rT)N(-d_2)$$

Secondary Greeks

- From just looking at equations or graphs it is clear that the greeks also have derivatives.
- E.g. the derivative of delta w.r.t. volatility, the derivative of Vega with respect to time etc.
- Many books and authors want you to think it is vital to know the exact form of all of these.
- Not really true.
- You need to know they *exist*.
- But no trading or risk management ideas are dependent on these actual equations.

Secondary Greeks

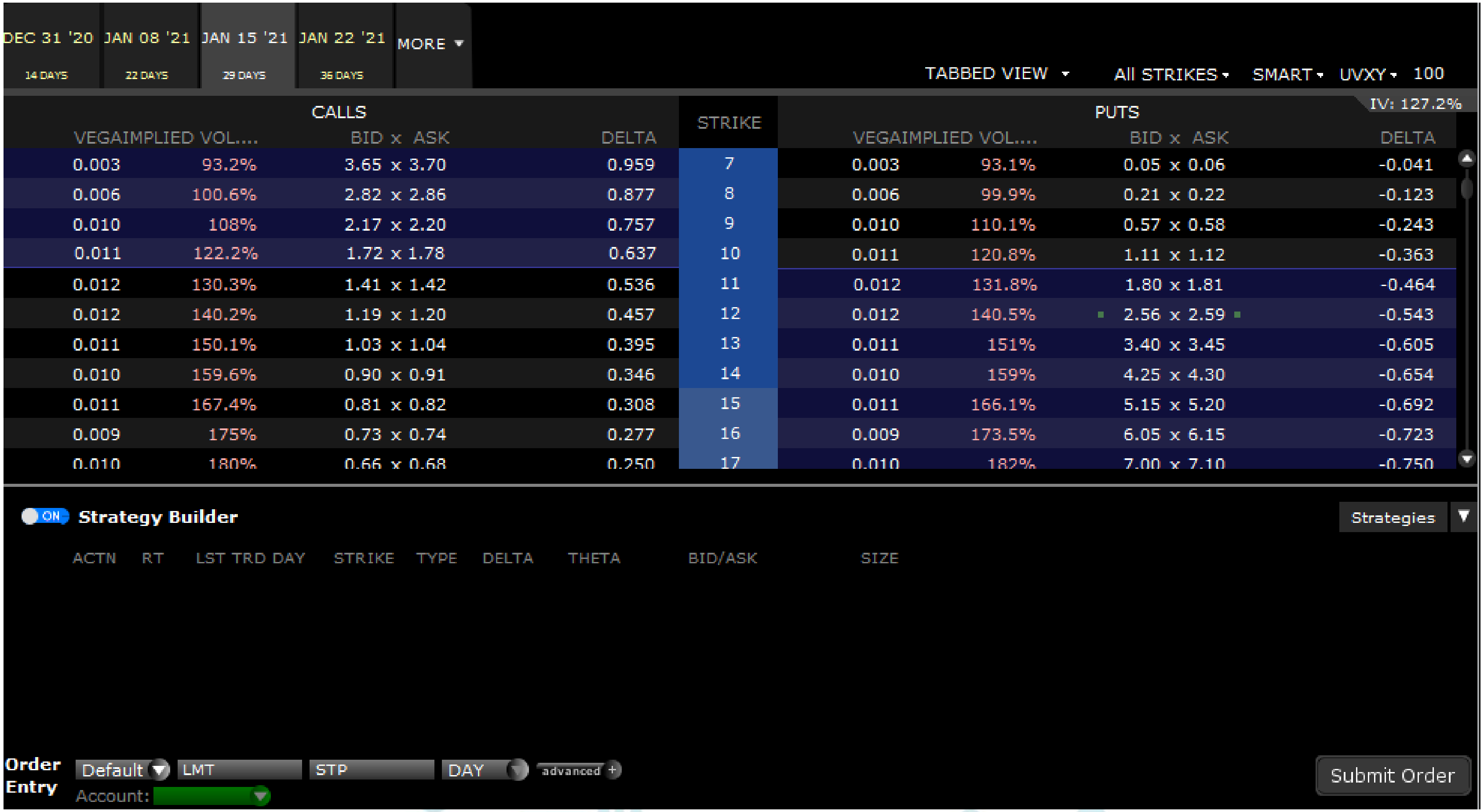
- Any professional level trading or risk system will show these greeks, but detailed understanding of their characteristics isn't really essential.

  (We will return to this when we look at risk management).
- B.T.W. There is no standard nomenclature for these derivatives.
- E.G. The derivative of delta with respect to volatility is variously called "vanna", "DdeltaDvol", "DdelV" or even "alpha", depending on who the trader learned from.

Greek Example

- Imagine these are my aggregate greeks:
- Long 5000 delta: I make $5000 if the stock goes up $1.
- Long 100 gamma: my delta changes to 5100 if the stock goes up a dollar.
- Short 500 theta: I expect to lose $500 a day due to lose of future optionality and carry costs.
- Long 1000 vega: if volatility goes up one point, I make $1000.

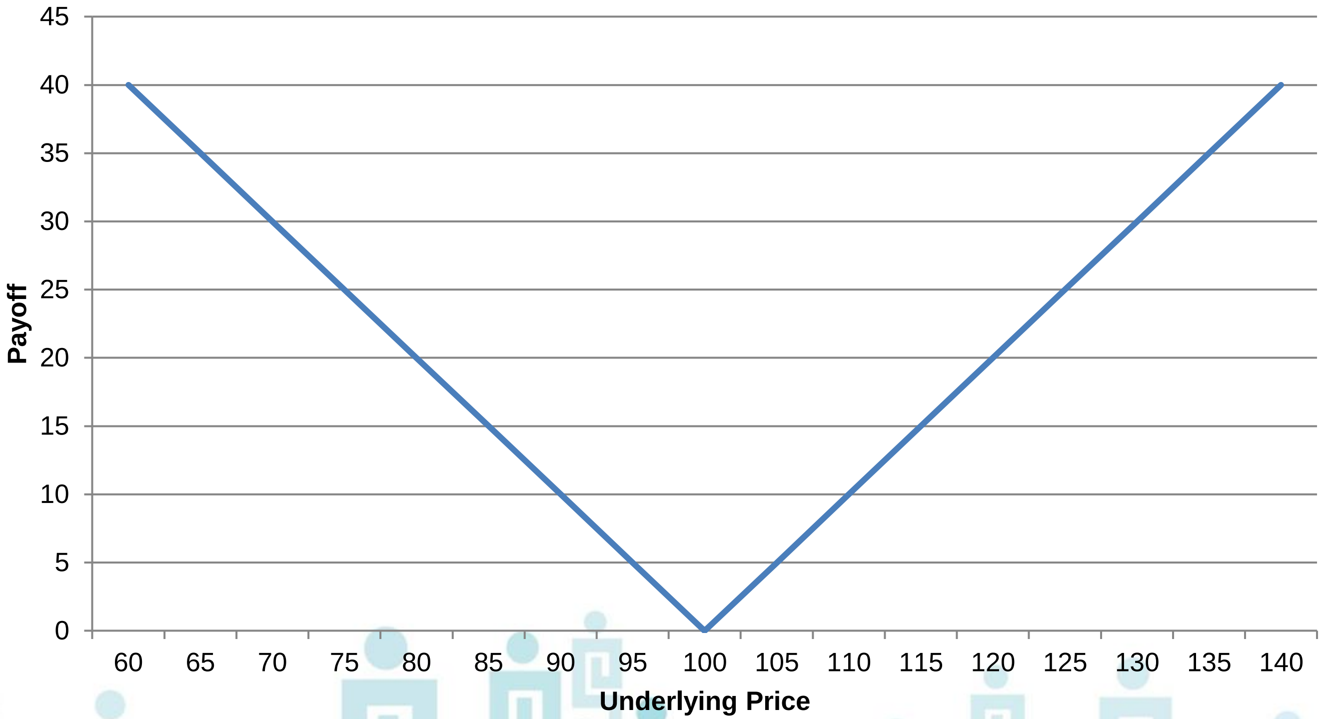
Trading Screen



v 10.1.0

Why Options?

- We have seen several examples of how options can be used to speculate on volatility.
- Straddle (or strangle, butterfly…)



Why Options?

- We have seen several examples of how options can be used to speculate on volatility.
- If we can make a bullish option position into a bearish option position by adding stock, then we should also be able to directionally neutralize an option.
- So, P/C parity is an important precursor to volatility trading.

Why Options?

- We have seen several examples of how options can be used to speculate on volatility.
- Black-Scholes-Merton (or binomial) model.
- Create a *hedged* portfolio by selling h shares short.

$$Portfolio = C - hS$$

- For a small move in the underlying, we want

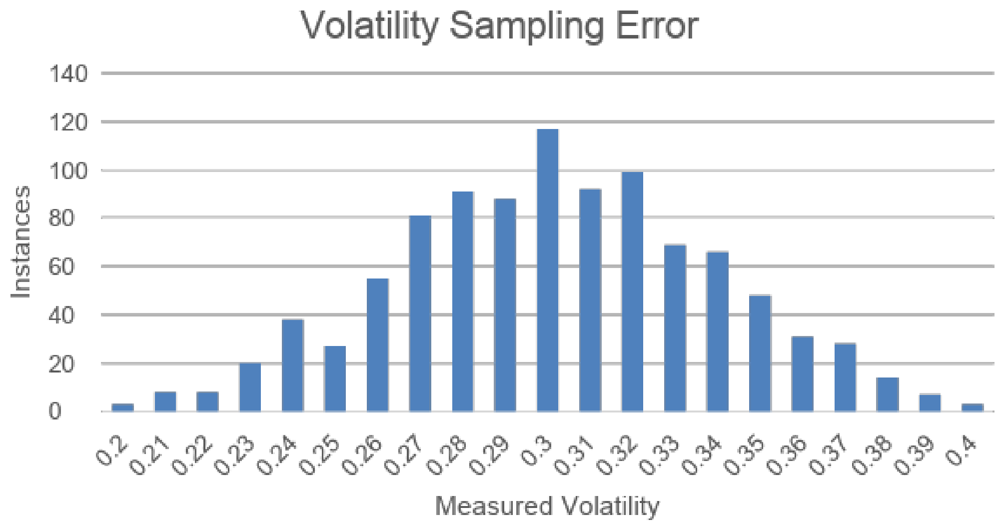
$$\frac{\partial Portfolio}{\partial S} = \frac{\partial C}{\partial S} - h = 0$$

Some Issues With Volatility

- What is N?
- There are an infinite number of volatilities.
- Smaller N => more responsive but more volatile measurement.

Sampling Error

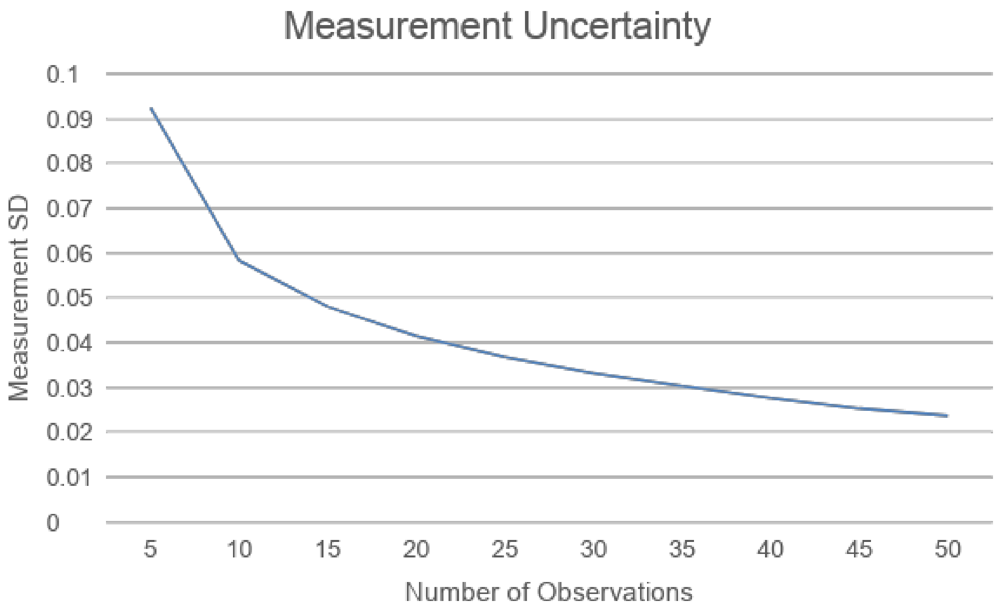
- Example: N=30, true volatility=0.3.



- Accurate but not very precise.
- Average: 0.30, SD: 0.04.
- (Based on a Monte-Carlo simulation with 1000 paths)

Sampling Error

- Error decreases with N.
- Measured value isn't necessarily a true value.



Alternatives

Volatility is **defined** as the standard deviation of the (log) returns. Equivalently variance is

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}\left(r_i - \bar{r}\right)^2$$

For a population, this is the MLE estimator. But we are dealing with a sample, not a population.

Are there better ways to estimate population volatility from sample volatility.

Maybe…

Parkinson Estimator

- Uses daily range.

$$\sigma = \sqrt{\frac{1}{4N\ln 2}\sum_{i=1}^{N}\left(\ln\frac{h_i}{l_i}\right)^2}$$

- This estimator has greater precision.
- This shouldn't be too surprising as our 30 days now gives us 60 data points.

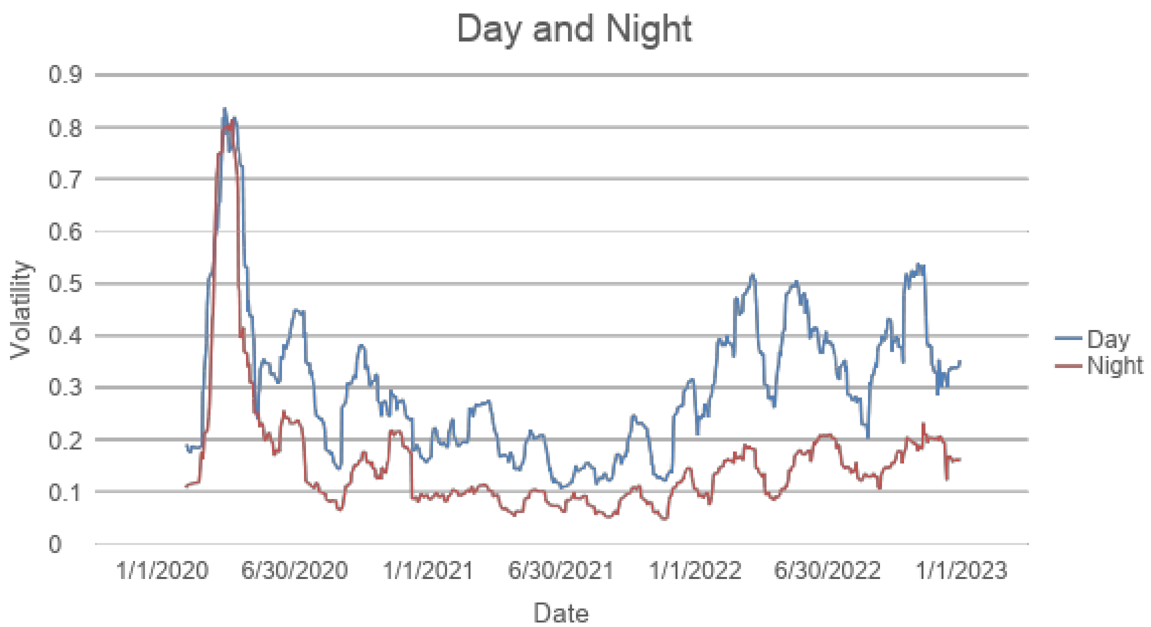
Parkinson Estimator: Problems

- First, this estimate is biased low.
- This is because the "true" extremes might be overnight when we aren't observing.

Days	Parkinson Vol/True Vol
10	0.81
20	0.86
50	0.91
100	0.93

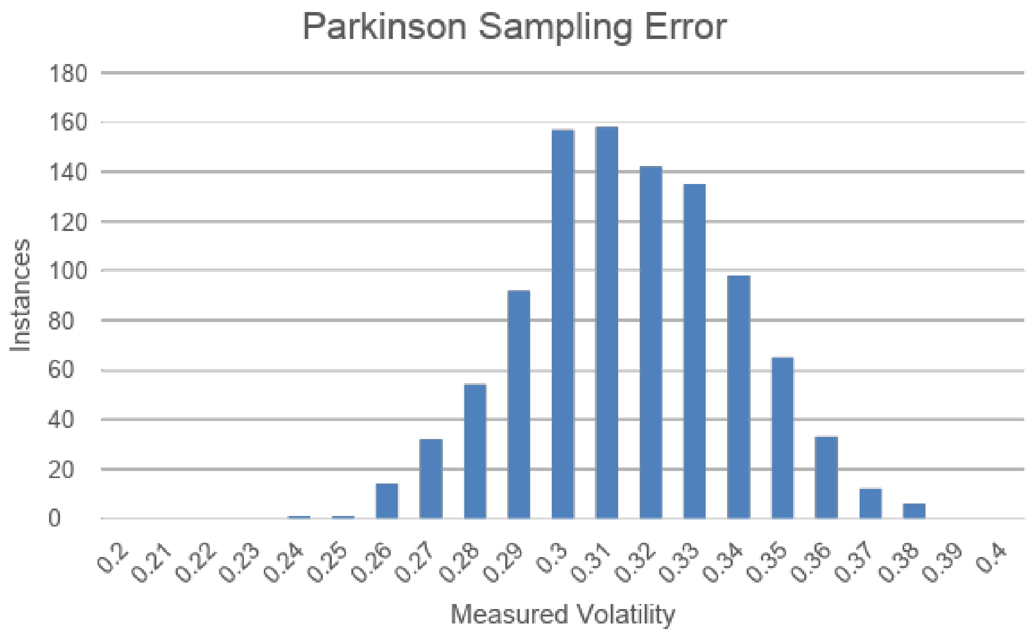
Parkinson Estimator: Problems

- Second, it assumes day and night volatility are the same.
- They aren't.



Parkinson Sampling Error

- Example: N=30, true volatility=0.5.



- Average: 0.30, SD: 0.023.

Many Other Estimators

- Parkinson: Uses high and low prices.
- Garman-Klass: Uses open, close, high and low.
- Rogers-Satchel: Includes drift.
- Yang-Zhang: Includes jumps.
- All get “better” for known, constant distributions.
- But this isn’t true of the market. Here we don’t know the true process or parameters. So, we can’t ever know which estimator is really best.

Common Misconceptions

- Some traders like range-based estimators “Because they measure what we do”.
- Others hate them “Because you can’t trade those prices”.
- Both are wrong.

A Misconception

- The best estimate of a variable is independent of how you intend to use it.
- Temperature is also a variable that needs to be measured.
- But, while people use the measurement for different purposes, they don't insist on an idiosyncratic measurement method.
- The brewer doesn't estimate temperature by fermentation times. She uses a thermometer.
- The best estimate is the best estimate and can be used for many different purposes.

Conclusion

- Use whatever you want.
- Choose something and stick with it.
- It is better to really know a model well than use a number haphazardly.
- I use an average of close to close and Parkinson.

Aside: High Frequency Data

- This data is now available and cheap, but there are other problems.
- If you sample too fast, you will get false volatility due to bid-ask bounce.
- Overnight jumps are a significant issue.

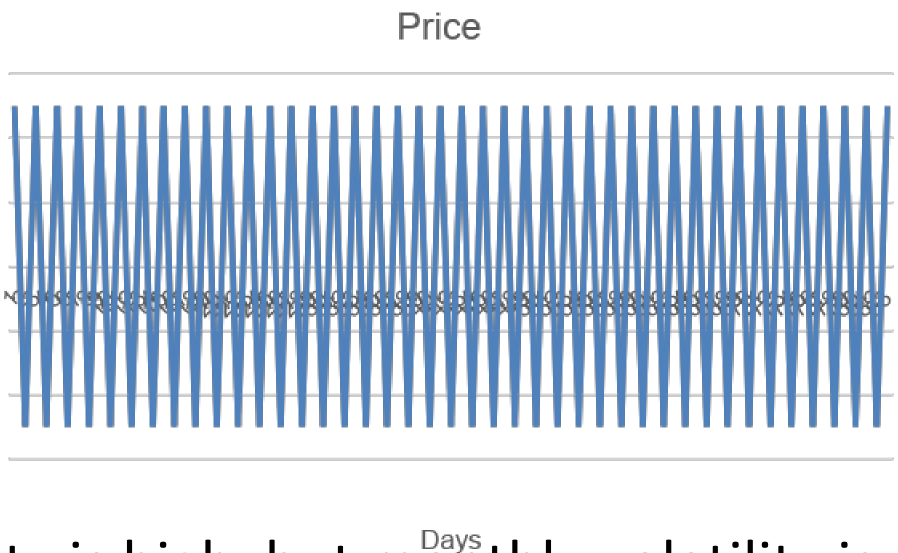
Overnight Jumps

- Treat these as a separate process. Use close to open range to estimate, $\sigma_{overnight}$.
- Use high frequency data to estimate $\sigma_{intraday}$.
- Combine these
- $\sigma_{total} = \sqrt{\frac{T_0}{T}\sigma^2_{overnight} + \frac{T_i}{T}\sigma^2_{intraday}}$

Where T_o/T is proportion of time the market is closed, and T_i/T is the time the market is open.

Aside: High Frequency Data

- Volatilities based on different sampling periods are *different things*.

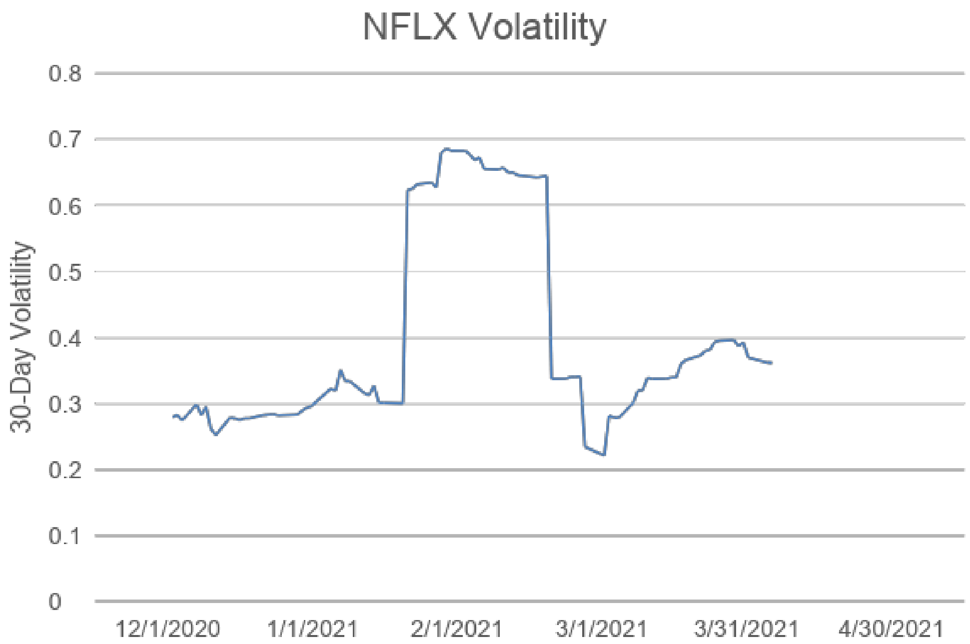


- Daily volatility is high, but monthly volatility is zero.
- Used as a test for mean reversion: the variance ratio test.

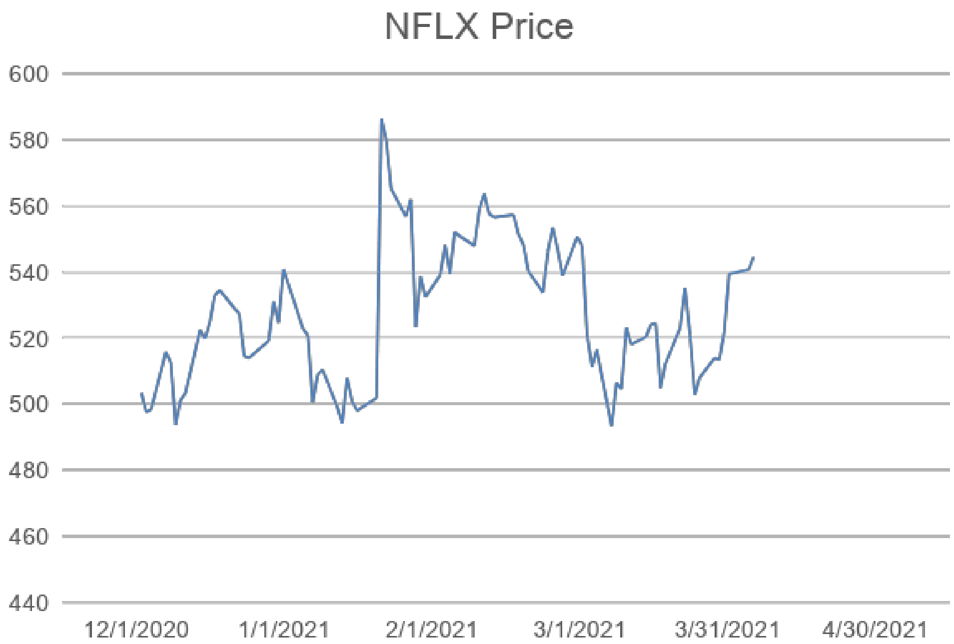
Forecasting Volatility

- After measuring what it IS, we need to forecast what it WILL BE.
- Assume next N days will be like last N. “Rolling window method”.
- This means large price jumps will be included in the volatility calculation for N days, then drop out.

Forecasting Volatility



Forecasting Volatility



Forecasting Volatility

	A	B	C	D	E	F	G
1	Date	Price	Return				
38	12/23/2020	514.48	-0.02467	0.283873			
39	12/24/2020	513.97	-0.00099	0.282172			
40	12/28/2020	519.12	0.00997	0.283186			
41	12/29/2020	530.87	0.022382	0.28905			
42	12/30/2020	524.59	-0.0119	0.293638			
43	12/31/2020	540.73	0.030303	0.296298			
44	1/4/2021	522.86	-0.03361	0.322412			
45	1/5/2021	520.8	-0.00395	0.319421			
46	1/6/2021	500.49	-0.03978	0.350951			
47	1/7/2021	508.89	0.016644	0.334075			
48	1/8/2021	510.4	0.002963	0.333691			
49	1/11/2021	499.1	-0.02239	0.315473			
50	1/12/2021	494.25	-0.00977	0.312775			
51	1/13/2021	507.79	0.027027	0.326901			
52	1/14/2021	500.86	-0.01374	0.300948			
53	1/15/2021	497.98	-0.00577	0.301048			
54	1/19/2021	501.77	0.007582	0.300111			
55	1/20/2021	586.34	0.155758	0.6232	=15.87*STDEV(C35:C55)		
56	1/21/2021	579.84	-0.01115	0.625563			
57	1/22/2021	565.17	-0.02563	0.632122			

Forecasting Volatility

- Weight more recent observations more heavily, so the effect of the jump subsides.
- For example, Exponentially Weighted Moving Average (EWMA).

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)r^2$$

- λ normally between 0.9 and 0.99.
- GARCH (family) adds a long-term level we expect mean reversion to.

Forecasting Volatility

Forecasting Volatility

- GARCH(1,1) is specified as:

$$\sigma_t^2 = \gamma V + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2$$

- Normally fit using maximum likelihood method.
- Needs about 1000 data points to give a stable set of values (four years of daily data).

GARCH - FORECASTING VOLATILITY

$$\sigma_{t+x}^2 = (1 - \alpha - \beta)V + \alpha r_{t+x-1}^2 + \beta \sigma_{t+x-1}^2$$

$$(\sigma_{t+x}^2 - V) = \alpha(r_{t+x-1}^2 - V) + \beta(\sigma_{t+x-1}^2 - V)$$

$$but\ E(r_t^2) = \sigma_t^2$$

$$E(\sigma_{t+x}^2 - V) = (\alpha + \beta)E(\sigma_{t+x-1}^2 - V)$$

$$E(\sigma_{t+x}^2) = V + (\alpha + \beta)^x E(\sigma_t^2 - V)$$

Forecasting Volatility

- The GARCH(1,1) likelihood function is:

$$\prod \frac{1}{\sqrt{2\pi\sigma_i^2}} exp\left(\frac{-r_i^2}{2\sigma_i^2}\right)$$

Forecasting Volatility

- But we usually use the log-likelihood because sums are easier to maximize numerically:

$$\sum\left[-ln(\sigma_i^2)-\frac{r_i^2}{\sigma_i^2}\right]$$

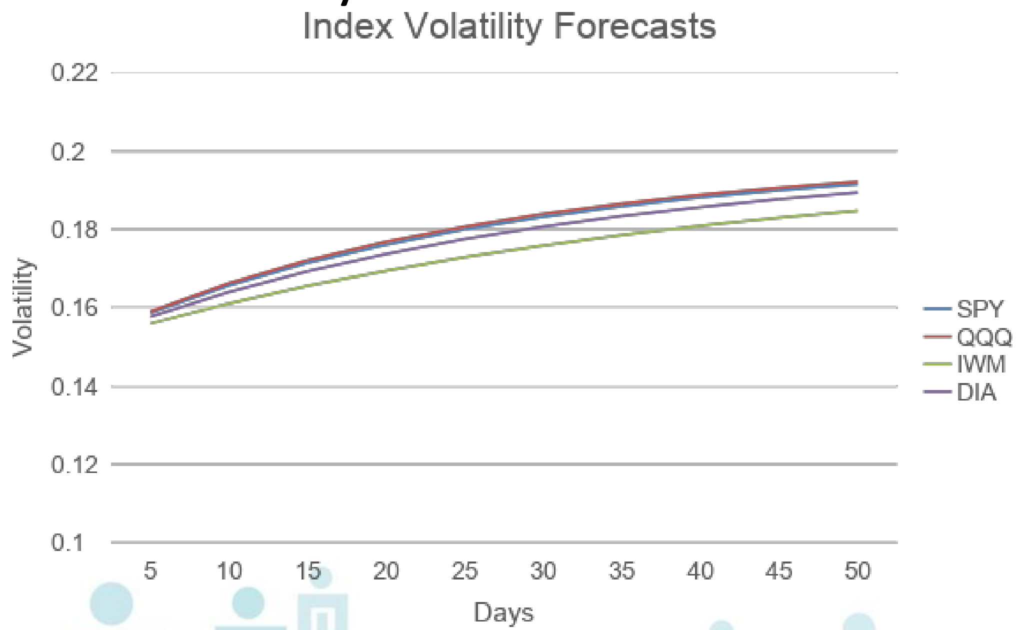
GARCH Parameters

- For 2017 to 2022:

Product	Gamma	Alpha	Beta
SPY	0.0328	0.2398	0.7274
QQQ	0.0341	0.1824	0.7834
IWM	0.0217	0.1421	0.8362
DIA	0.0285	0.2236	0.7479

GARCH Parameters

- These may look very different, but the resulting forecasts are very similar.



Forecasting Volatility

- "Trader GARCH" exogenously specifies the parameters just as we do with EWMA.
- Set V to a long-term value of variance.
- Have alpha somewhere around 0.01 to 0.1.
- Have beta between 0.9 and 1.0.
- Choose gamma so parameters sum to one.

Context

If I tell you a sports team had a score of 14, what do you think?
Your first question should be "What sport?"
In cricket, 14 is terrible.
In soccer, it is amazing.
In golf, it is impossible.
All measurements need to be placed in context.

Forecasts in Context

- Look at the RANGE of volatility, not just current value.
- E.g. Measure volatility for periods of 20, 40, 60, 80 and 100 days then calculate quartiles (NFLX, 2016-2021).

	20 Day Vol	40 Day Vol	60 Day Vol	80 Day Vol	100 Day Vol
maximum	0.89	0.73	0.63	0.59	0.56
75th Percentile	0.47	0.47	0.45	0.46	0.45
Median	0.33	0.37	0.39	0.39	0.38
25th Percentile	0.27	0.29	0.32	0.33	0.33
Minimum	0.14	0.15	0.18	0.19	0.20