4 BASICS OF PROBABILITY THEORY

In this chapter we will discuss the building blocks of probability theory. It is a worthwhile effort to get acquainted with the concepts and common terms to lay a strong foundation for the upcoming chapters.

4.1 BASIC DEFINITIONS & TERMS

Its Christmas holiday season and Sophie has decided to take a break from her trading. She sits down with some friends to play a game of Ludo. In this game a six-sided fair die is rolled repeatedly to determine how the Ludo pieces will be moved on the board.

However, before rolling the die, the result is not known. This would be an instance of a **random experiment**. To be specific, a random experiment is a process by which we observe something uncertain.

After the experiment is over, the result is known. An **outcome** is a result of a random experiment. When a fair die is rolled, there are six possible outcomes - we can get any one of the numbers 1 to 6 with equal likelihood.

The set of all possible outcomes is called the **sample space**. Thus, in Sophie's case of rolling a fair die, the sample space **Ω,** would look like:

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Similarly, if the random experiment was to toss a fair coin one time, the sample space would look like:

$\Omega = \{\text{Heads, Tails}\}$

When we repeat a random experiment several times, we call each one of them a **trial**. Thus, a trial is a particular instance of a random experiment. In the example of tossing a coin, each trial will result in either heads or tails.

Any subset of the sample space is called an **event.** For example, getting an odd number when we roll a fair die is an event and it can be represented as:

E = {1, 3, 5}

More often than not our aim is to find the probability of events associated with random experiments. The **probability** of an event is a value representing the chance that the event will take place. An event that is certain to happen has a probability of 1. An event that cannot possibly happen has a probability of zero. If there is a chance that an event will happen, then its probability is between zero and 1.

The probability of an event can be mathematically stated as:

Probability of an event = P (event) = $\frac{number\ of\ favorable\ outcomes}{Total\ number\ of\ possible\ outcomes}$

Thus, the probability of getting an odd number when a fair die is rolled (event E) is given by:

$$\mathrm{P\ (E)} = \frac{number\ of\ favorable\ outcomes}{Total\ number\ of\ possible\ outcomes} = \frac{3}{6} = 0.5$$

4.2 UNION & INTERSECTION

If A and B are events, the union of A and B is represented by A∪B, which is also an event and occurs if **at least one** of A **or** B occurs.

Similarly, the intersection of both A and B occurs when A **and** B occur together and is represented by A∩B.

For example, consider the following two events:

E: Getting an odd number when a fair die is rolled, i.e.
E= {1, 3, 5}

F: Getting a number divisible by 3 when a fair die is rolled i.e.
F= {3, 6}

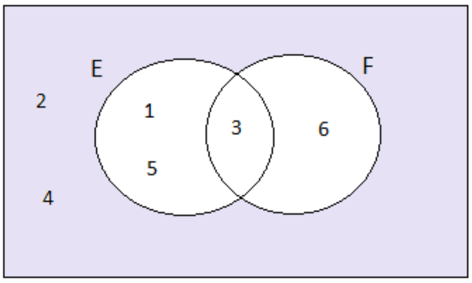
Thus, three out of the six possible outcomes are favorable for event E, and hence

Probability of event E = P(E) = $\frac{3}{6} = 0.5$

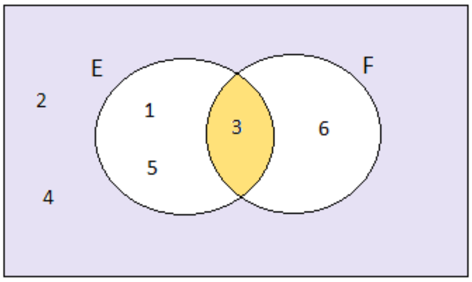
Probability of event F = P(F) = $\frac{2}{6} = 0.333$

Let us look at the Venn diagram to get a pictorial representation:

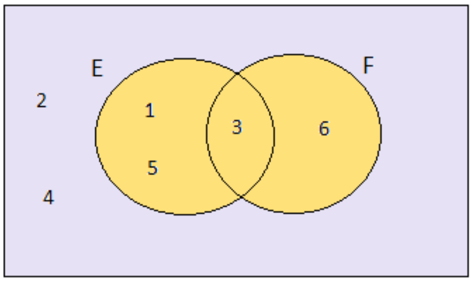
Events E and F



Intersection



Union



The event **$E \cap F$** occurs only if we get 3 when we roll the die. In set notation,
$E \cap F = \{3\}$

As only one out of the six outcomes is favorable for event **$E \cap F$,**

$P(E \cap F) = \frac{1}{6}$

Similarly, the event **$E \cup F$** occurs if we get any outcome from E or F i.e. if we get a 1 or 3 or 5 or 6 when we roll the die. In set notation,
$E \cup F = \{1, 3, 5, 6\}$

As four out of the six outcomes are favorable for event **$E \cup F$,**

$P(E \cup F) = \frac{4}{6}$

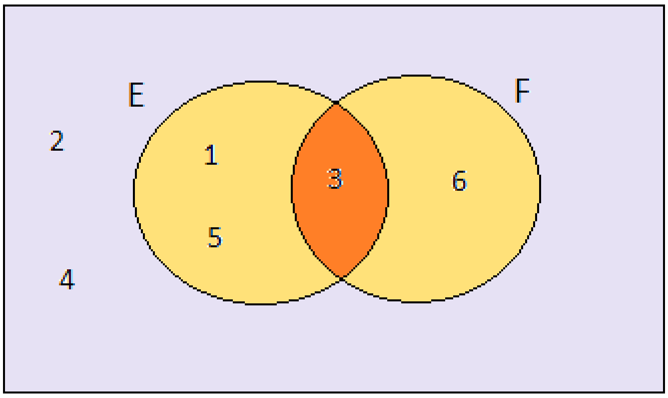
Extending this idea to multiple sets, if $E_1, E_2, E_3, \ldots, E_n$, are n events, then the event **$E_1 \cup E_2 \cup E_3 \ldots \cup En$** occurs if **at least one** of the n events occurs. Similarly, the event **$E_1 \cap E_2 \cap E_3 \ldots\ldots \cap En$** occurs, if **all** of the n events occur. The probabilities for these events can be calculated in accordance with the number of favorable outcomes as well.

Notation: For any two events A and B,
$P(A \cap B) = P(A \text{ and } B) = P(A,B)$

$P(A \cup B) = P(A \text{ or } B)$

4.3 INCLUSION- EXCLUSION PRINCIPLE

Let us look at the events E and F discussed in the previous section once again. We know that
$E = \{1, 3, 5\}$
$F = \{3, 6\}$
$E \cap F = \{3\}$



Note that when we calculated the probability of event **E∪F**, we counted the intersection of E and F (outcome 3) only once to avoid double counting.
$(E \cup F) = \{1, 3, 5, 6\}$

In general, if A and B are two events then
$P(A \cup B) = P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$

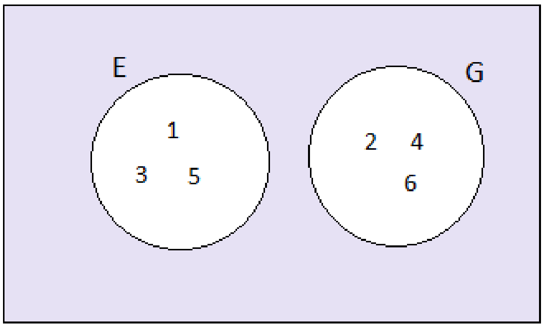
4.4 MUTUALLY EXCLUSIVE OR DISJOINT EVENTS

Now consider the following two events:
E: Getting an odd number when we roll a fair die, i.e.
$E = \{1, 3, 5\}$

G: Getting an even number when we roll a fair die, i.e.
$G = \{2, 4, 6\}$

We know that in a single trial of rolling a fair die, we can either get an even number or an odd number but not both. Such events are called mutually exclusive or disjoint events.

Events E and G are mutually exclusive



The intersection of mutually exclusive events is a null set i.e. an event containing no value and consequently its probability is zero.

$E \cap G$ = A null/empty set = {}

$P(E \cap G) = \frac{0}{6} = 0$

In general, if A and B are two mutually exclusive events then
$P(A \cup B) = P(A \text{ or } B) = P(A) + P(B)$
&
$P(A \cap B) = 0$

4.5 Conditional Probability

Let us consider events E and F again.

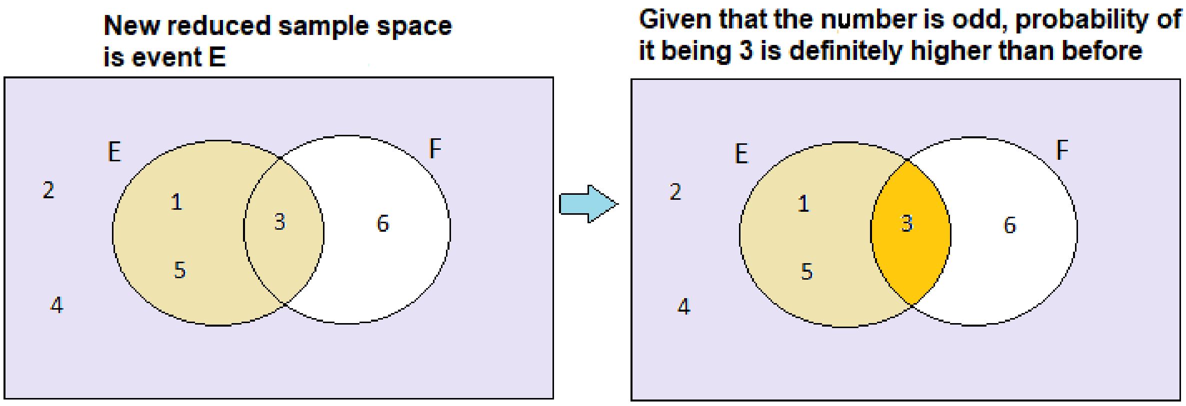
E: Getting an odd number when a fair die is rolled, i.e.
E= {1, 3, 5}

F: Getting a number divisible by 3 when a fair die is rolled i.e.
F= {3, 6}

Suppose a player rolls a die and gets an odd number. She tells her friends that it is an odd number but does not reveal exactly which number it is. What is the chance that it is a 3, given that the number obtained is odd?

In other words, we are asking the probability of event F, conditional on event E having occurred. This is often denoted using "|" symbol. Thus, the probability of number being '3' given that the number is definitely odd is represented by P**(F | E).**

If no information was given then the probability of getting a '3' is just 1/6. However, now we have additional information that the number is odd. This leads to a reduced set of possibilities i.e. a reduced sample space given by {1, 5, 3}, which is event E.



Thus, the probability of the number being '3', given that the outcome is odd is the ratio of probability for the event E∩F to the probability for event E.

$$P(\text{'3'}|\text{ odd}) = P(F \mid E) = \frac{1/6}{3/6} = \frac{1}{3}$$

In general, if A and B are two events then the conditional probability of B given A is:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

4.6 Independent Events

Let us consider the following two events when we roll a fair die two times (two trials):

Event J: Getting a '4' in the first trial
Event K: Getting a '4' in the second trial

As the die is a fair one, we know that:

$$P(J) = \frac{1}{6} = P(K)$$

That is, the events J and K are completely random and do not depend on each other in any way. Such events are called **independent events**.

The probability of getting a '4' in the first trial followed by a '4' again in the second trial is nothing but the product of the two individual probabilities.

$$P(K \mid J) = P(K) = \frac{1}{6}$$

In general, two events A and B are said to be independent if the probability of occurrence of one of them is not affected by the occurrence of the other, i.e.

P (A | B) = P(A) and P(B|A) = P(B)

4.7 BAYES' THEOREM

In the Bayesian approach we update the probability of an event based on the new information that is obtained. The previous probability is referred to as the *prior* probability and the updated probability is called the *posterior* probability.

Bayes' theorem is an important way of calculating conditional probabilities. It establishes an important relationship between the conditional probabilities of two events.

In general, if A and B are two events then according to Bayes' theorem, the conditional probability of B given A is given by:

$$P(B \mid A) = \frac{P(A|B) * P(B)}{P(A)}$$

For example, consider the following two events:

John is on the verge of winning the game and needs a number greater than or equal to '4' to win. Let us denote this event by F, i.e.

F: Getting an outcome greater than or equal to '4' when a fair die is rolled i.e.
F= {4, 5, 6}

Initially, without any other information, **P (John wins the game) = P(F) = $\frac{1}{3}$.**

The die is rolled. Before John can see the outcome Lilly mischievously covers the die with her palm and only reveals that the outcome is an even number. How does this new information change the probability of John winning the game?

Let us denote the event of getting an even number by G.

G: Getting an even outcome when a fair die is rolled, i.e.
G= {2, 4, 6}

Thus, P(G) = $\frac{1}{3}$ and P(G | F) = P(even outcome | outcome greater than 4) = $\frac{2}{3}$.

Using the Bayes' theorem, the updated probability of John winning the game can be calculated as:

$$P(F \mid G) = \frac{P(G|F) * P(F)}{P(G)} = \frac{\frac{2}{3} * \frac{1}{3}}{\frac{1}{3}} = \frac{2}{3}$$

Thus, given the new information, the chances of John winning the game have doubled!